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Comparison of reduction in formal decision contexts

Jinhai Li [a,*], Cherukuri Aswani Kumar [b], Changlin Mei [c], Xizhao Wang [d]

[a] Faculty of Science, Kunming University of Science and Technology, Kunming 650500, Yunnan, PR China
[b] School of Information Technology and Engineering, VIT University, Vellore 632014, India
[c] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, Shaanxi, PR China
[d] College of Computer Science and Software Engineering, Shenzhen University, Shenzhen 518060, PR China

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ABSTRACT

In formal concept analysis, many reduction methods have recently been proposed for formal decision contexts, and each of them was to reduce formal decision contexts with a particular purpose. However, little attention has been paid to the comparison of their differences from various aspects. In fact, this problem is very important because it can provide evidence to select an appropriate reduction method for a given specific case. To address this problem, our study mainly focuses on clarifying the relationship among the existing reduction methods in formal decision contexts. Firstly, we give a rule-based review of the existing reduction methods, revealing the type of rules that each of them can preserve. Secondly, we analyze the relationship among the consistencies introduced by the existing reduction methods. More specifically, Wei's first consistency (see [39]) is stronger than others, while her second one is weaker than the remainder except Wu's consistency (see [43]). Finally, we make a comparison of the existing reductions, concluding that Li's reduction (see [14]) maintaining the non-redundant decision rules of a formal decision context is coarser than others. The results obtained in this paper are beneficial for users to select an appropriate reduction method for meeting their requirements.

1. Introduction

Formal concept analysis (FCA) was proposed by Wille [41] to restructure lattice theory based on hierarchies of formal concepts. To be more specific, formal concepts together with a predefined subconcept–superconcept relation form a complete lattice, called concept lattice [8]. After 34 years of development, FCA has been successfully applied in many fields [5,7,31,42,44,51].

However, with the deepening of the research on FCA, many scholars have found that the constraints on Wille's concept lattice are too restrictive for many applications. In order to loosen these constraints, several types of generalized concept lattices were developed such as power concept lattice [10], rough concept lattice [12], three-way concept lattice [27], AFS concept lattice [35], object-oriented concept lattice [45,46] and approximate concept lattice [16]. Moreover, some studies have also been made on the relationship among the generalized concept lattices [22,37]. In addition, how to reduce these generalized concept lattices was discussed from the perspectives of matrix factorization [2], fuzzy K-means clustering [3], rough set [20], covering-based rough set [18], and lattice isomorphism [19,24,50]. Besides, mining implications or association rules based on the generalized concept lattices has also attracted much attention in recent years [1,9,21,34,47].

* Corresponding author. Fax: +86 0871 65916703.
E-mail addresses: jhlixjtu@163.com (J. Li), aswanis@gmail.com (Ch. Aswani Kumar), clmei@mail.xjtu.edu.cn (C. Mei), xzwang@szu.edu.cn (X. Wang).

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Table 1
A formal context for describing the characteristics of the reduction techniques.

	Characteristics					
	CL is needed	Consistency is needed	For general dataset	Output all reducts	Information loss	Heuristic
Wei et al. [39] (a)	×	×		×		
Wei et al. [39] (b)	×	×		×	×	
Wang & Zhang [38]	×	×		×	×	
Wu et al. [43]		×		×		
Li et al. [13]	×	×			×	×
Li et al. [14]	×		×	×		

In order to implement a special decision making with FCA, Zhang and Qiu [49] introduced the notion of a formal decision context by dividing the attributes of a formal context into two parts: conditional attributes and decision attributes. Then, decisions in formal decision contexts were made by all kinds of rules, such as decision implications [28], granular rules [43], and decision rules [13,14,30]. In fact, formal decision contexts can be viewed as an extension of formal contexts. It is of practical importance to make such an extension. The reasons are summarized as follows: (1) it is often encountered in the real world that relational databases include decision attributes; (2) formal decision contexts allow us to give a more particular explanation of the induced implications since certain types of rules can be derived from a formal decision context to meet users' different requirements; (3) formal decision contexts can help us to deal with the basic issues in rough set theory [25] from the viewpoint of FCA, such as lower and upper approximation operators [33], reduction [6] and attribute dependency [11] in decision tables. The third reason is in accordance with the fact that formal decision contexts are beneficial to the comparison and combination of FCA with rough set theory [40].

As far as we know, knowledge reduction is one of the most important issues in the study of formal decision contexts, and its aim is to avoid the redundancy of attributes while preserving the predefined properties [36,48]. Up to now, many reduction methods (e.g. [13,14,26,29,38,39,43]) have been obtained. For example, Wei et al. [39] put forward two consistencies between the concept lattice of the formal context (G, M, I) and that of (G, N, J), where G is an object set, M is a conditional attribute set, and N is a decision attribute set of a formal decision context. More specifically, the first consistency means that for any concept of (G, N, J), there exists a concept of (G, M, I) such that their extents are the same. The second consistency means that for any concept of (G, N, J), there exists a concept of (G, M, I) such that the extent of the latter concept is included in the extent of the former one. Then, based on the defined consistencies, two reduction methods were developed for consistent formal decision contexts. More precisely, the first reduction method is to avoid the redundancy of attributes while preserving the non-redundant decision rules, but the second one only preserves the number of decision rules. Wang and Zhang [38] presented a different one whose aim is to avoid the redundancy of attributes while preserving the number of feedforward non-redundant decision rules. Moreover, Wu et al. [43] proposed a novel reduction method whose aim is to avoid the redundancy of attributes while preserving the granular rules. In addition, Li et al. [13] designed another reduction method whose aim is to avoid the redundancy of attributes while preserving the number of non-redundant decision rules. Note that the above five reduction methods are only suitable for consistent formal decision contexts. In other words, they will become invalid for inconsistent formal decision contexts. Motivated by this problem, Li et al. [14] gave a new reduction method which can be suitable for general formal decision contexts, and they used it to derive more compact decision rules from a formal decision context. In fact, the existing reduction methods can be classified, according to different types of concept lattices on which they are based, into three categories: the first category was performed by Wille's concept lattice (e.g. [13,14,38,39,43]), the second by the concept lattice induced by axialities (e.g. [26]), and the third by the object-oriented and property-oriented concept lattices (e.g. [29]). Since the second and third categories were obtained by extending the results of the first category into the generalized concept lattices, it is sufficient to discuss the reduction methods in the first category (i.e., the six reduction methods in [13,14,38,39,43] whose characteristics are summarized in Table 1, where CL is the abbreviation of "concept lattice", and a and b respectively represent the first and second reduction methods in [39]) when we study the existing reduction methods.

Although many reduction methods have been proposed for formal decision contexts, how to select a suitable one for a given specific case still needs to be studied. Up to now, little work has been devoted to this problem. In fact, to achieve this task, it is necessary to clarify the inherent relationship among the existing reduction methods. Our paper is going to investigate the relationship among the reduction methods in formal decision contexts from three aspects: providing a rule-based review of them, comparing their consistencies, and making a comparison of their reductions. The first aspect reveals the type of rules that each of them can preserve, and the other two can help users to select an appropriate reduction method to meet their requirements.

The rest of this paper is organized as follows. Section 2 recalls some basic notions related to formal concept analysis and introduces several types of rules in formal decision contexts. Section 3 provides a rule-based review of the reduction methods in formal decision contexts. Section 4 makes a comparison of the consistencies defined in the existing reduction methods. Section 5 investigates the relationship among the reductions under the five kinds of consistent formal decision contexts. The paper is then concluded with a brief summary and an outlook for further research.

2. Preliminaries

In this section, we review some basic notions related to formal concept analysis and introduce several types of rules in formal decision contexts.

2.1. Formal contexts and concept lattices

Definition 1 *([41])*. A formal context is a triplet (G, M, I), where G is a set of objects, M is a set of attributes, and I is a binary relation on the Cartesian product $G \times M$ with $(g, m) \in I$ indicating that the object g has the attribute m, and $(g, m) \notin I$ indicating the opposite.

For a formal context (G, M, I), Wille defined a pair of concept forming operators: for any $A \subseteq G$ and $B \subseteq M$,

$$A' = \{m \in M \mid \forall g \in A, (g, m) \in I\},$$
$$B' = \{g \in G \mid \forall m \in B, (g, m) \in I\}.$$

Definition 2 *([41])*. Let (G, M, I) be a formal context, $A \subseteq G$ and $B \subseteq M$. The pair (A, B) is called a formal concept if $A' = B$ and $B' = A$. In this case, A and B are called the extent and the intent of the formal concept (A, B), respectively.

Concepts of a formal context (G, M, I) together with the following subconcept–superconcept relation

$$(A_1, B_1) \leq (A_2, B_2) \Longleftrightarrow A_1 \subseteq A_2$$

form a complete lattice, called the concept lattice of (G, M, I) [8]. Hereinafter, we denote it by $\underline{\mathfrak{B}}(G, M, I)$ in which the infimum ($\wedge$) and the supremum ($\vee$) of two concepts are respectively defined by

$$(A_1, B_1) \wedge (A_2, B_2) = \left(A_1 \cap A_2, (B_1 \cup B_2)''\right),$$
$$(A_1, B_1) \vee (A_2, B_2) = \left((A_1 \cup A_2)'', B_1 \cap B_2\right).$$

Let (G, M, I) be a formal context and $E \subseteq M$. The restriction of I on the Cartesian product $G \times E$, denoted by I_E, is defined as:

$$(g, m) \in I_E \Longleftrightarrow (g, m) \in I \text{ for any } (g, m) \in G \times E.$$

Then the triplet (G, E, I_E) is a subcontext of (G, M, I) [37]. Of course, concept forming operators on the subcontext (G, E, I_E) can be defined in a similar way. That is, for any $A \subseteq G$ and $B \subseteq E$,

$$A'^E = \{m \in E \mid \forall g \in A, (g, m) \in I_E\},$$
$$B'^E = \{g \in G \mid \forall m \in B, (g, m) \in I_E\}.$$

Besides, we denote the concept lattice of (G, E, I_E) by $\underline{\mathfrak{B}}(G, E, I_E)$.

Proposition 1 *([41]). Let (G, M, I) be a formal context and $E \subseteq M$. For $A, A_1, A_2 \subseteq G$ and $B, B_1, B_2 \subseteq E$, the following statements hold:*

(i) $A_1 \subseteq A_2 \Rightarrow A_2'^E \subseteq A_1'^E$, $B_1 \subseteq B_2 \Rightarrow B_2'^E \subseteq B_1'^E$;
(ii) $A \subseteq A'^E{}'^E$, $B \subseteq B'^E{}'^E$;
(iii) $(A'^E{}'^E, A'^E), (B'^E, B'^E{}'^E) \in \underline{\mathfrak{B}}(G, E, I_E)$.

Note that $(g'^E{}'^E, g'^E), (m'^E, m'^E{}'^E)$ are referred to as the object and the attribute concepts, respectively. In other words, it is the special case of (iii) of Proposition 1 by taking $A = \{g\}$ and $B = \{m\}$.

For a specific purpose, we say that an extent A of (G, E, I_E) is minimal to include g if there does not exist another extent A_0 such that $g \in A_0 \subset A$. Then, we have the following proposition.

Proposition 2. *Let (G, E, I_E) be a subcontext of (G, M, I) and $g \in G$. Then $g'^E{}'^E$ is a minimal extent of (G, E, I_E) including g.*

2.2. Several types of rules in formal decision contexts

In this section, we introduce the decision rules, feedforward non-redundant decision rules, feedback non-redundant decision rules, and granular rules in formal decision contexts.

Definition 3 *([49])*. A formal decision context is a quintuple (G, M, I, N, J), where (G, M, I) and (G, N, J) are formal contexts and their attributes are disjoint. We call M the conditional attribute set and N the decision attribute set of (G, M, I, N, J). In this paper, we suppose that the formal decision contexts to be discussed are regular. That is, for each of objects, it has at least one attribute but not all attributes; besides, for each of attributes, it is owned by at least one object but not all objects.

Hereinafter, we denote the concept lattice of (G, N, J) by $\underline{\mathfrak{B}}(G, N, J)$, and we call (G, E, I_E, N, J) a subcontext of (G, M, I, N, J). In other words, a subcontext (G, E, I_E, N, J) is obtained by removing some conditional attributes from the original formal decision context.

To facilitate our subsequent discussion, two binary relations α_E and β_E on the Cartesian product $\underline{\mathfrak{B}}(G, E, I_E) \times \underline{\mathfrak{B}}(G, N, J)$ are defined as follows: for $(A, B) \in \underline{\mathfrak{B}}(G, E, I_E)$ and $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$,

$$\alpha_E((A,B),(C,D)) = \begin{cases} 1, & \text{if } A \subseteq C \text{ and } \forall (A_0, B_0) \in \underline{\mathfrak{B}}(G, E, I_E),\ A \subset A_0 \Rightarrow A_0 \nsubseteq C, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_E((A,B),(C,D)) = \begin{cases} 1, & \text{if } A \subseteq C \text{ and } \forall (C_0, D_0) \in \underline{\mathfrak{B}}(G, N, J),\ C_0 \subset C \Rightarrow A \nsubseteq C_0, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4 *([13,14])*. Let (G, M, I, N, J) be a formal decision context and $E \subseteq M$. For any $(A, B) \in \underline{\mathfrak{B}}(G, E, I_E)$ and $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $A, B, C, D \neq \emptyset$, if $A \subseteq C$, we say that a decision rule, denoted by $B \to D$, can be induced by the formal concepts (A, B) and (C, D). We call B the premise and C the conclusion of the decision rule $B \to D$.

Moreover, we denote by $\mathcal{R}_E$ the set of all the decision rules induced by the formal concepts in $\underline{\mathfrak{B}}(G, E, I_E)$ and those in $\underline{\mathfrak{B}}(G, N, J)$. In fact, the decision rule $B \to D$ means "If $\wedge B$, then $\wedge D$". That is, it is a conjunctive rule. In addition, its support is defined as

$$\mathrm{Supp}(B \to D) = \frac{|B^{\prime_E}|}{|G|}.$$

Definition 5 *([15])*. Let (G, M, I, N, J) be a formal decision context, $E \subseteq M$ and $B \to D \in \mathcal{R}_E$.

(i) $B \to D$ is called a feedforward redundant decision rule in $\mathcal{R}_E$ if $\beta_E((A, B), (C, D)) = 0$; otherwise, it is a feedforward non-redundant decision rule in $\mathcal{R}_E$.
(ii) $B \to D$ is called a feedback redundant decision rule in $\mathcal{R}_E$ if $\alpha_E((A, B), (C, D)) = 0$; otherwise, it is a feedback non-redundant decision rule in $\mathcal{R}_E$.
(iii) $B \to D$ is said to be redundant in $\mathcal{R}_E$ if it is a feedforward or feedback redundant decision rule in $\mathcal{R}_E$; otherwise, it is non-redundant in $\mathcal{R}_E$.

For convenience of description, we denote by $\mathcal{R}_E^*$, $\mathcal{R}_E^{f*}$, $\mathcal{R}_E^{b*}$ the sets of all the non-redundant, feedforward non-redundant, and feedback non-redundant decision rules of $\mathcal{R}_E$, respectively.

Definition 6 *([43])*. Let (G, M, I, N, J) be a formal decision context and $E \subseteq M$. $B \to D \in \mathcal{R}_E$ is called a granular rule of $\mathcal{R}_E$ if there exists $g \in G$ such that $B = g^{\prime_E}$ and $D = g^{\prime_N}$, where $\prime_N$ represents the concept forming operator $\prime$ on (G, N, J).

It can be known from Definitions 4 and 6 that a granular rule is a special decision rule induced by the object concepts $(g^{\prime_E \prime_E}, g^{\prime_E})$ and $(g^{\prime_N \prime_N}, g^{\prime_N})$.

Definition 7 *([14])*. Let (G, M, I, N, J) be a formal decision context and $E \subseteq M$. We say that $B \to D \in \mathcal{R}_E$ can be implied by $B_0 \to D_0 \in \mathcal{R}_E$ if $B_0 \subseteq B$ and $D \subseteq D_0$.

Proposition 3. *Let (G, M, I, N, J) be a formal decision context and $E \subseteq M$. Then $B \to D \in \mathcal{R}_E$ is non-redundant in $\mathcal{R}_E$ if and only if there does not exist another decision rule $B_0 \to D_0 \in \mathcal{R}_E$ such that $B_0 \to D_0$ implies $B \to D$.*

Proof. It is immediate from Definitions 5 and 7. □

Definition 8 *([14])*. Let (G, M, I, N, J) be a formal decision context and $E \subseteq F \subseteq M$. We say that $\mathcal{R}_F$ can be implied by $\mathcal{R}_E$, denoted by $\mathcal{R}_E \Rightarrow \mathcal{R}_F$, if each decision rule of $\mathcal{R}_F$ can be implied by one of $\mathcal{R}_E$.

It can be seen from Definition 8 that if $\mathcal{R}_E \Rightarrow \mathcal{R}_M$, then the subcontext (G, E, I_E, N, J) can preserve the decision rule information derived from (G, M, I, N, J) although some attributes have been removed from the original formal decision context.

3. A rule-based review of the reduction methods in formal decision contexts

In this section, six reduction methods are going to be reviewed from the perspective of rule acquisition in order to reveal the kind of rules that each of them can preserve. Note that the order we review them below is based on the year they were published. For convenience, we denote them by Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a, and Li-2011b.

Before embarking on reviewing the six reduction methods, it deserves to be mentioned that all of them are to avoid the redundancy of attributes while preserving the predefined properties. That is to say, the reduction ideas of these six reduction methods are the same, and their main differences lie in what properties they want to preserve. So, it is sufficient to introduce the properties that these reductions need to preserve when reviewing them.

3.1. The Wei-2008a reduction method in formal decision contexts

The Wei-2008a reduction method, presented by Wei et al. [39] in 2008, is based on I-consistency. Concretely, a formal decision context $K = (G, M, I, N, J)$ is I-consistent if and only if for any concept $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$, there exists $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$ such that $A = C$. This kind of reduction is performed by I-reduct E which is a minimal subset of M such that (G, E, I_E, N, J) is I-consistent. In other words, its objective is to avoid the redundancy of attributes while preserving I-consistency.

Note that $E \subseteq M$ is called a I-consistent set if (G, E, I_E, N, J) is I-consistent. Then, a I-reduct of K is a I-consistent set such that it is minimal with respect to I-consistency.

Proposition 4. *Let $K = (G, M, I, N, J)$ be a formal decision context. If K is I-consistent, then every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can and only can generate one non-redundant decision rule $B \to D$ in $\mathcal{R}_M$.*

Proof. For every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$, there exists $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$ such that $A = C$ due to the I-consistency of K. Thus, we obtain $\alpha_M((A, B), (C, D)) = \beta_M((A, B), (C, D)) = 1$. Since K is regular, it follows $A, B \neq \emptyset$. By Definition 5, $B \to D$ is non-redundant in $\mathcal{R}_M$. The uniqueness of $B \to D$ is obvious. □

Theorem 1. *Let $K = (G, M, I, N, J)$ be a I-consistent formal decision context. Then $E \subseteq M$ is a I-reduct of K if and only if E is a minimal subset of M such that every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can and only can generate one non-redundant decision rule $B_0 \to D$ in $\mathcal{R}_E$ with $Supp(B_0 \to D) = Supp(B \to D)$, where $B \to D$ is the non-redundant decision rule generated by (C, D) in $\mathcal{R}_M$.*

Proof. Necessity. If $E \subseteq M$ is a I-reduct of K, then E is a minimal subset of M such that (G, E, I_E, N, J) is I-consistent. Similar to Proposition 4, we can prove that every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can and only can generate one non-redundant decision rule $B_0 \to D$ in $\mathcal{R}_E$. Without loss of generality, we assume that $B \to D$ is the non-redundant decision rule generated by (C, D) in $\mathcal{R}_M$. Note that there exist $(A_0, B_0) \in \underline{\mathfrak{B}}(G, E, I_E)$ and $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$ such that $|A_0| = |C| = |A|$, which leads to $\mathrm{Supp}(B_0 \to D) = \mathrm{Supp}(B \to D)$.

Sufficiency is trivial. □

It can be known from Theorem 1 that the Wei-2008a reduction method can preserve the non-redundant decision rule information derived from a I-consistent formal decision context.

3.2. The Wei-2008b reduction method in formal decision contexts

The Wei-2008b reduction method, also introduced by Wei et al. [39] in 2008, is based on a weaker consistency called II-consistency. Specifically, a formal decision context $K = (G, M, I, N, J)$ is said to be II-consistent if there exists an injection $f : \underline{\mathfrak{B}}(G, N, J) \to \underline{\mathfrak{B}}(G, M, I)$ such that the extent of each $f(C, D)$ is included in that of (C, D). This reduction is performed by a II-reduct E which is a minimal subset of M such that (G, E, I_E, N, J) is II-consistent. In other words, its aim is to avoid the redundancy of attributes while preserving II-consistency.

Note that $E \subseteq M$ is called a II-consistent set if (G, E, I_E, N, J) is II-consistent. Then, a II-reduct of K is a II-consistent set such that it is minimal with respect to II-consistency.

Proposition 5. *Let $K = (G, M, I, N, J)$ be a formal decision context. If K is II-consistent, then all the formal concepts $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate decision rules in $\mathcal{R}_M$ with their premises being pairwise different.*

Proof. Since K is II-consistent, there exists an injection $f : \underline{\mathfrak{B}}(G, N, J) \to \underline{\mathfrak{B}}(G, M, I)$ such that the extent of each $f(C, D)$ is included in that of (C, D). So, for any $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$, there exists $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$ such that $B \to D$ is a decision rule of $\mathcal{R}_M$. Moreover, the injection f can guarantee that the premises of these induced decision rules are pairwise different. □

Theorem 2. *Let $K = (G, M, I, N, J)$ be a II-consistent formal decision context. Then $E \subseteq M$ is a II-reduct of K if and only if E is a minimal subset of M such that all the formal concepts $(C, D) \in \underline{\mathcal{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate decision rules in $\mathcal{R}_E$ with their premises being pairwise different.*

Proof. It is trivial. □

It can be observed from Theorem 2 that the Wei-2008b reduction method can preserve the number of decision rules of a II-consistent formal decision context with their conclusions being pairwise different as well as their premises.

3.3. The Wang-2008 reduction method in formal decision contexts

The Wang-2008 reduction method, proposed by Wang and Zhang [38] in 2008, is based on III-consistency. To be more specific, a formal decision context $K = (G, M, I, N, J)$ is said to be III-consistent if for any $(C, D) \in \underline{\mathcal{B}}(G, N, J)$, there exists $(A, B) \in \underline{\mathcal{B}}(G, M, I)$ such that $\beta_M((A, B), (C, D)) = 1$. This reduction is performed by a III-reduct E which is a minimal subset of M such that (G, E, I_E, N, J) is III-consistent. In other words, its aim is to avoid the redundancy of attributes while preserving III-consistency.

Note that $E \subseteq M$ is called a III-consistent set if (G, E, I_E, N, J) is III-consistent. Then, a III-reduct of K is a III-consistent set such that it is minimal with respect to III-consistency.

Proposition 6. *Let $K = (G, M, I, N, J)$ be a formal decision context. If K is III-consistent, then every $(C, D) \in \underline{\mathcal{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate a feedforward non-redundant decision rule in $\mathcal{R}_M$.*

Proof. It is immediate from Definition 5. □

Theorem 3. *Let $K = (G, M, I, N, J)$ be a III-consistent formal decision context. Then $E \subseteq M$ is a III-reduct of K if and only if E is a minimal subset of M such that every $(C, D) \in \underline{\mathcal{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate a feedforward non-redundant decision rule in $\mathcal{R}_E$.*

Proof. It is trivial based on Proposition 6. □

It can be known from Theorem 3 that the Wang-2008 reduction method can preserve the number of feedforward non-redundant decision rules of a III-consistent formal decision context with their conclusions being pairwise different as well as their premises.

3.4. The Wu-2009 reduction method in formal decision contexts

The Wu-2009 reduction method, proposed by Wu et al. [43] in 2009, is based on IV-consistency. It is described as follows: a formal decision context $K = (G, M, I, N, J)$ is said to be IV-consistent if $g^{\prime_M \prime_M} \subseteq g^{\prime_N \prime_N}$ for any $g \in G$. That is, K is IV-consistent if and only if the conditional object granule $g^{\prime_M \prime_M}$ is smaller than the decision object granule $g^{\prime_N \prime_N}$.

This reduction is performed by a IV-reduct E which is a minimal subset of M such that (G, E, I_E, N, J) is IV-consistent. In other words, its objective is to avoid the redundancy of attributes while preserving IV-consistency. Note that $E \subseteq M$ is called a IV-consistent set if (G, E, I_E, N, J) is IV-consistent. Then, a IV-reduct of K is a IV-consistent set such that it is minimal with respect to IV-consistency.

Proposition 7. *Let $K = (G, M, I, N, J)$ be a formal decision context. If K is IV-consistent, then every $g \in G$ can lead to a granular rule $g^{\prime_M} \to g^{\prime_N}$ in $\mathcal{R}_M$.*

Proof. Since K is IV-consistent, then $g^{\prime_M \prime_M} \subseteq g^{\prime_N \prime_N}$ for any $g \in G$. Noting that K is regular, we conclude that $g^{\prime_M}$, $g^{\prime_M \prime_M}$, $g^{\prime_N}$ and $g^{\prime_N \prime_N}$ are all nonempty. As a result, $g^{\prime_M} \to g^{\prime_N}$ is a granular rule in $\mathcal{R}_M$. □

Theorem 4. *Let $K = (G, M, I, N, J)$ be a IV-consistent formal decision context. Then $E \subseteq M$ is a IV-reduct of K if and only if E is a minimal subset of M such that every $g \in G$ can lead to a granular rule $g^{\prime_E} \to g^{\prime_N}$ in $\mathcal{R}_E$.*

Proof. It is immediate from Proposition 7. □

By Definition 7, it is easy to verify that the granular rule $g^{\prime_E} \to g^{\prime_N}$ can imply $g^{\prime_M} \to g^{\prime_N}$ for any $g \in G$. Thus, based on Theorem 4, the Wu-2009 reduction method can preserve the granular rule information derived from a IV-consistent formal decision context since every granular rule of $\mathcal{R}_M$ can be implied by one of $\mathcal{R}_E$.

3.5. The Li-2011a reduction method in formal decision contexts

The Li-2011a reduction method, presented by Li et al. [13] in 2011, is based on V-consistency. Formally, a formal decision context $K = (G, M, I, N, J)$ is said to be V-consistent if every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can lead to a non-redundant decision rule $B_0 \to D$ in $\mathcal{R}_M$ with $\text{Supp}(B_0 \to D) = \max_{B \to D \in \mathcal{R}_M^*} \{\text{Supp}(B \to D)\}$.

This reduction is performed by a V-reduct E which is a minimal subset of M such that (G, E, I_E, N, J) is V-consistent. In other words, its aim is to avoid the redundancy of attributes while preserving V-consistency. Note that $E \subseteq M$ is called a V-consistent set if (G, E, I_E, N, J) is V-consistent. Then, a V-reduct of K is a V-consistent set such that it is minimal with respect to V-consistency.

In summary, the Li-2011a reduction method can preserve the number of non-redundant decision rules of a V-consistent formal decision context with their conclusions being pairwise different and their supports being maximal with respect to those induced by the same decision concept.

3.6. The Li-2011b reduction method in formal decision contexts

The Li-2011b reduction method, proposed by Li et al. [14] in 2011, does not depend on any kind of consistencies but was established from the viewpoint of preserving the decision rule information derived from a formal decision context. Its main purpose is to derive more compact decision rules from a formal decision context $K = (G, M, I, N, J)$, and it is performed via a VI-reduct. To be more concrete, a VI-reduct E is a minimal subset of M such that $\mathcal{R}_E \Rightarrow \mathcal{R}_M$ (or equivalently $\mathcal{R}_E^* \Rightarrow \mathcal{R}_M^*$). In addition, a subset E of M is called a VI-consistent set of K if $\mathcal{R}_E \Rightarrow \mathcal{R}_M$.

Note that the Li-2011b reduction of a formal decision context will lead to a smaller data set because some unnecessary attributes are removed. Moreover, it will obtain more compact decision rules since some attributes in the premises of decision rules can be reduced. Besides, considering that every non-redundant decision rule derived from the original formal decision context can be induced by a decision rule derived from the reduced formal decision context, we conclude that this reduction keeps the non-redundant decision rule information.

4. Comparison of consistencies

In Section 3, we have reviewed six reduction methods in formal decision contexts from the viewpoint of rule acquisition. It was found that the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009 and Li-2011a reduction methods are only suitable for consistent formal decision contexts, while the Li-2011b reduction method is suitable for general formal decision contexts. Note that general formal decision contexts can be divided into two categories: consistent formal decision contexts and inconsistent formal decision contexts. Then, the above reduction methods except Li-2011b will become invalid for inconsistent formal decision contexts. So, it is not appropriate to compare them in inconsistent formal decision contexts.

In this paper, our idea is to compare the six reduction methods under certain type of consistent formal decision contexts, and the consistency is one of them introduced in Section 3. Although we have known that the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009 and Li-2011a reduction methods are suitable for their respective consistent formal decision contexts, it is still necessary to clarify whether they are also suitable for other types of consistent formal decision contexts. For solving this problem, it seems less effective if we verify them one by one. The reason is that each reduction method except Li-2011b should be checked whether it is suitable for other four types of consistent formal decision contexts. In other words, we need to check them 20 times. In fact, there is a more effective way to achieve this task. It can be done with the help of the strong and weak relations among these consistencies. Detailedly speaking, if a reduction method is suitable for a consistent formal decision context, then it must also be suitable for a weaker consistent formal decision context. So, it is more effective to solve this problem based on the strong and weak relations among the five consistencies.

In what follows, we discuss the strong and weak relations among the five consistencies.

Definition 9. Let $i, j \in \{I, II, III, IV, V\}$. If any j-consistent formal decision context is i-consistent, we say that i-consistency is weaker than j-consistency, or equivalently j-consistency is stronger than i-consistency, denoted by i-consistency $\leq$ j-consistency. If i-consistency $\leq$ j-consistency and j-consistency $\leq$ i-consistency, we say that i-consistency is equivalent to j-consistency, denoted by i-consistency $=$ j-consistency. Moreover, if i-consistency $\leq$ j-consistency and i-consistency $\neq$ j-consistency, we say that i-consistency is strictly weaker than j-consistency, denoted by i-consistency $<$ j-consistency.

Theorem 5. *For I, II, III, IV and V-consistencies, the following statements hold:*

(1) *II-consistency $\leq$ III-consistency $=$ V-consistency $\leq$ I-consistency;*
(2) *IV-consistency $\leq$ I-consistency.*

Proof. (1) In order to prove it, we need to show

(a) II-consistency $\leq$ III-consistency,

Table 2
A formal decision context $K = (G, M, I, N, J)$.

G	a	b	c	d	e	f	g	d_1	d_2	d_3	d_4	d_5
1		×	×					×				×
2	×								×			
3		×		×		×		×		×	×	
4	×								×			
5		×		×	×			×		×		
6		×		×		×	×	×		×	×	

(b) III-consistency $\leq V$-consistency,
(c) V-consistency $\leq III$-consistency,
(d) V-consistency $\leq I$-consistency.

First, we prove (a). For any III-consistent formal decision context $K = (G, M, I, N, J)$, it follows from Proposition 6 that every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate a feedforward non-redundant decision rule in $\mathcal{R}_M$. Then, for any two different concepts $(C_1, D_1), (C_2, D_2) \in \underline{\mathfrak{B}}(G, N, J)$ with $C_1, D_1, C_2, D_2 \neq \emptyset$, there exist $(A_1, B_1), (A_2, B_2) \in \underline{\mathfrak{B}}(G, M, I)$ such that both $B_1 \to D_1$ and $B_2 \to D_2$ are feedforward non-redundant decision rules. Moreover, (A_1, B_1) and (A_2, B_2) are different due to $(C_1, D_1) \neq (C_2, D_2)$. As a result, K is II-consistent. Based on Definition 9, (a) is at hand.

Second, we prove (b). For any V-consistent formal decision context K, we know that every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can lead to a non-redundant decision rule $B \to D$ in $\mathcal{R}_M$. Of course, every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can lead to a feedforward non-redundant decision rule in $\mathcal{R}_M$. That is, K is III-consistent. So, the proof of (b) is completed.

Third, we prove (c). For any III-consistent formal decision context K, it follows from Proposition 6 that every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate a feedforward non-redundant decision rule in $\mathcal{R}_M$. Of course, we can select a feedforward non-redundant decision rule $B_0 \to D$ such that $\mathrm{Supp}(B_0 \to D) = \max_{B \to D \in \mathcal{R}_M^*} \{\mathrm{Supp}(B \to D)\}$. Moreover, it is easy to show that the selected decision rule $B_0 \to D$ is non-redundant in $\mathcal{R}_M$. So, K is V-consistent, and hence (c) is proved.

Finally, we prove (d). For any I-consistent formal decision context K, it can be known from Proposition 4 that every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can and only can generate one non-redundant decision rule in $\mathcal{R}_M$. Undoubtedly, every $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate a non-redundant decision rule $B_0 \to D$ in $\mathcal{R}_M$ with $\mathrm{Supp}(B_0 \to D) = \max_{B \to D \in \mathcal{R}_M^*} \{\mathrm{Supp}(B \to D)\}$. That is to say, K is V-consistent, and then (d) is at hand.

(2) For any I-consistent formal decision context K, we know from Section 3.1 that for any $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$, there exists $(A, B) \in \underline{\mathfrak{B}}(G, M, I)$ such that $A = C$. Particularly, for object concept $(g'^{N'N}, g'^{N}) \in \underline{\mathfrak{B}}(G, N, J)$, there exists $(A_0, B_0) \in \underline{\mathfrak{B}}(G, M, I)$ such that $A_0 = g'^{N'N}$. Based on Proposition 2, $g'^{M'M}$ is the minimal extent including the object g. Moreover, considering that g also belongs to A_0, we conclude $g'^{M'M} \subseteq A_0$. As a result, $g'^{M'M} \subseteq g'^{N'N}$. So, K is IV-consistent, and the proof is completed. □

It can be seen from Theorem 5 that III-consistency and V-consistency are equivalent to each other. So, in what follows, we classify them into the same category and choose III-consistency as a representative when they are discussed. Moreover, in order to clarify the overall relationship among the five consistencies, it is still required to check whether or not the following statements are true:

(1) I-consistency $\leq II$-consistency;
(2) I-consistency $\leq III$-consistency;
(3) I-consistency $\leq IV$-consistency;
(4) II-consistency $\leq IV$-consistency;
(5) III-consistency $\leq II$-consistency;
(6) III-consistency $\leq IV$-consistency;
(7) IV-consistency $\leq II$-consistency;
(8) IV-consistency $\leq III$-consistency.

In fact, none of the above statements is true. The following is a counterexample to confirm our assertion.

Example 1. Table 2 depicts a formal decision context $K = (G, M, I, N, J)$, where $G = \{1, 2, 3, 4, 5, 6\}$, $M = \{a, b, c, d, e, f, g\}$ and $N = \{d_1, d_2, d_3, d_4, d_5\}$.

By the concept lattice construction algorithm in [23] (i.e. AddIntent), the Hasse diagrams of the concept lattices of the formal contexts (G, M, I) and (G, N, J) are shown in Figs. 1 and 2, respectively. For brevity, a nonempty object (or attribute) set in the figures is denoted only by listing its elements in sequence. For example, $\{3, 5, 6\}$ and $\{b, d\}$ are simply denoted by 356 and bd, respectively.

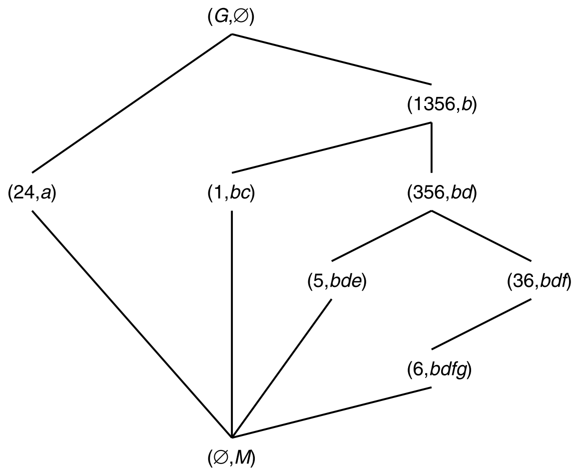

Fig. 1. $\underline{\mathfrak{B}}(G, M, I)$.

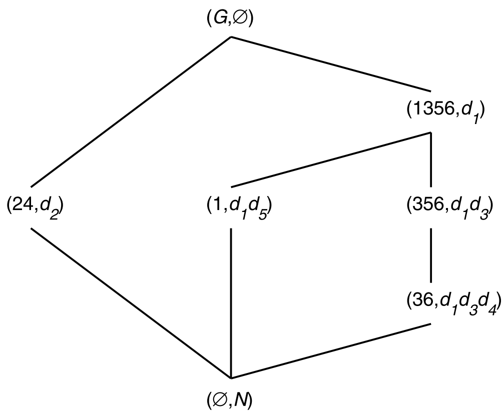

Fig. 2. $\underline{\mathfrak{B}}(G, N, J)$.

First, let $E = \{a, b, c, e, f\}$. Then Fig. 3 depicts the Hasse diagram of the concept lattice of the subcontext (G, E, I_E). From Figs. 2 and 3, we know that $K_E = (G, E, I_E, N, J)$ is II, III, IV-consistent. However, K_E is not I-consistent since for the concept $(356, d_1d_3) \in \underline{\mathfrak{B}}(G, N, J)$, there does not exist $(A, B) \in \underline{\mathfrak{B}}(G, E, I_E)$ such that $A = \{3, 5, 6\}$. Thus,

(1) I-consistency $\leq II$-consistency,
(2) I-consistency $\leq III$-consistency,
(3) I-consistency $\leq IV$-consistency

are not true.

Second, take $F = \{a, c, e, f\}$. The concept lattice of the subcontext (G, F, I_F) is given in Fig. 4. Then it is easy to see that $K_F = (G, F, I_F, N, J)$ is not II-consistent. However,

$$\begin{aligned}
&1^{'_F'_F} = \{1\} = 1^{'_N'_N},\\
&2^{'_F'_F} = \{2, 4\} = 2^{'_N'_N},\\
&3^{'_F'_F} = \{3, 6\} = 3^{'_N'_N},\\
&4^{'_F'_F} = \{2, 4\} = 4^{'_N'_N},\\
&5^{'_F'_F} = \{5\} \subset \{3, 5, 6\} = 5^{'_N'_N},\\
&6^{'_F'_F} = \{3, 6\} = 6^{'_N'_N}.
\end{aligned}$$

That is, K_F is IV-consistent. As a result,

(4) II-consistency $\leq IV$-consistency

does not hold.

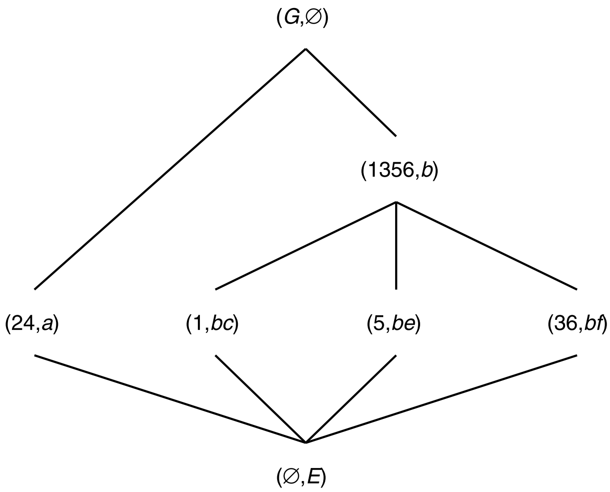

Fig. 3. $\underline{\mathcal{B}}(G, E, I_E)$.

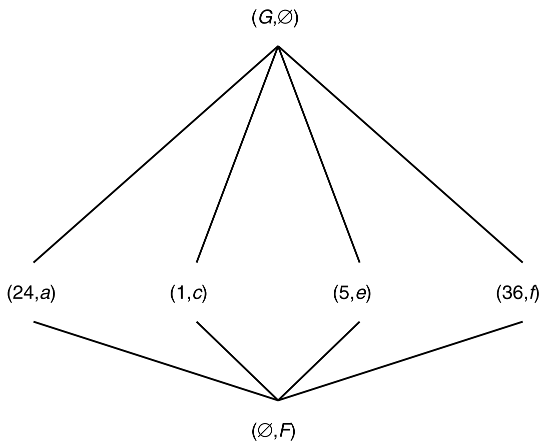

Fig. 4. $\underline{\mathcal{B}}(G, F, I_F)$.

Third, let $S = \{a, c, e, f, g\}$. Then Fig. 5 depicts the concept lattice of the subcontext (G, S, I_S). It is easy to verify that $K_S = (G, S, I_S, N, J)$ is not III-consistent since the concept $(1356, d_1) \in \underline{\mathcal{B}}(G, N, J)$ cannot lead to a feedforward non-redundant decision rule in $\mathcal{R}_S$. However, K_S is II-consistent and IV-consistent. Thus,

(5) III-consistency $\leq II$-consistency,
(6) III-consistency $\leq IV$-consistency

are not true.

Finally, take $T = \{a, b, c, e, g\}$. The Hasse diagram of the concept lattice of the subcontext (G, T, I_T) is shown in Fig. 6. Note that

$$3^{\prime T \prime T} = \{1, 3, 5, 6\} \nsubseteq \{3, 6\} = 3^{\prime N \prime N}.$$

Then, $K_T = (G, T, I_T, N, J)$ is not IV-consistent. But, it is easy to check that K_T is II-consistent and III-consistent. So,

(7) IV-consistency $\leq II$-consistency,
(8) IV-consistency $\leq III$-consistency

are not true.

Based on the above discussion, the relationship among I, II, III, IV and V-consistencies can be described by Fig. 7 in which the consistency becomes weaker from top to bottom.

Now, we are ready to give an answer to the question raised at the beginning of this section. That is, it is necessary to clarify whether or not the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reduction methods are

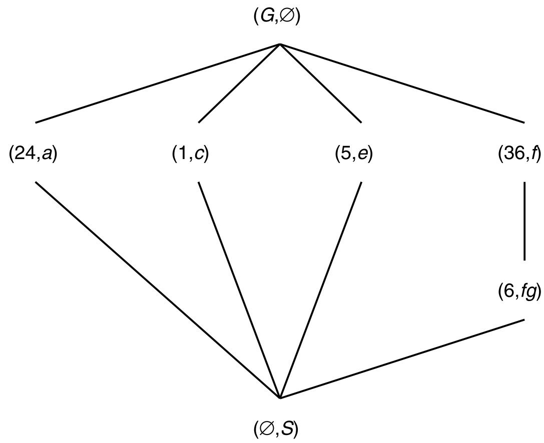

Fig. 5. $\underline{\mathfrak{B}}(G, S, I_S)$.

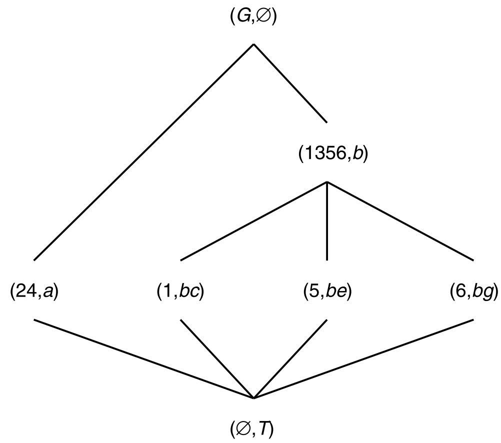

Fig. 6. $\underline{\mathfrak{B}}(G, T, I_T)$.

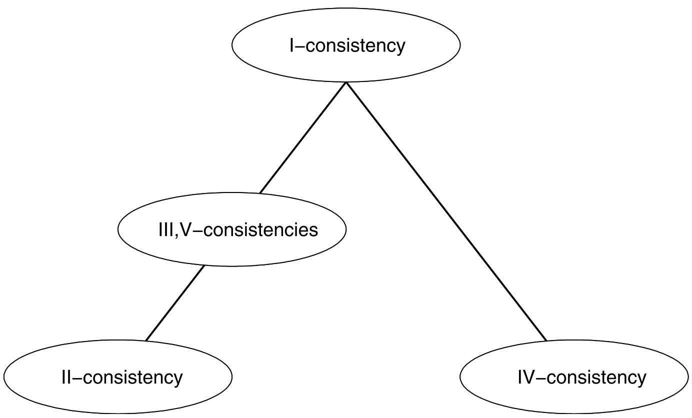

Fig. 7. The relationship among I, II, III, IV and V-consistencies.

suitable for I, II, III, IV and V-consistent formal decision contexts. In what follows, we are going to use a concept lattice to describe it.

The scenario can be described as follows: Note that if a reduction method is suitable for a consistent formal decision context, then it must also be suitable for a weaker one. Based on Fig. 7, we can create a table (e.g. Table 3) to show whether or not the six reduction methods are suitable for the five types of consistent formal decision contexts. In the table, r_1, r_2, r_3, r_4, r_5 and r_6 are respectively used to represent the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b

Table 3
A formal context (G, M, I).

G	I	II	III	IV	V
r_1	×				
r_2	×	×	×		×
r_3	×		×		×
r_4	×			×	
r_5	×		×		×
r_6	×	×	×	×	×

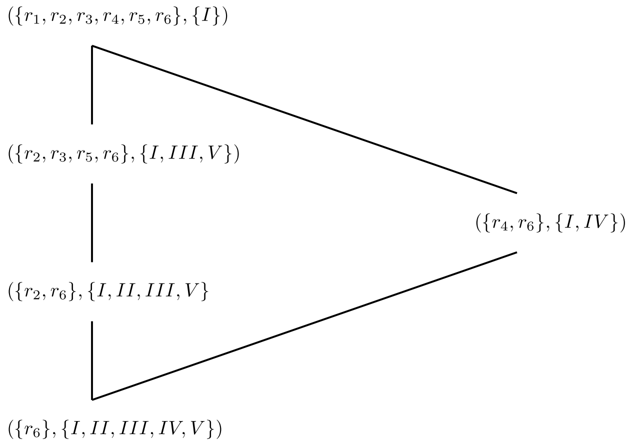

Fig. 8. A concept lattice for describing the reduction methods.

reduction methods, and I, II, III, IV and V are the abbreviations of I, II, III, IV and V-consistent formal decision contexts. Moreover, the cross × indicates that the reduction method in its row is suitable for the type of consistent formal decision contexts in its column.

Then, the Hasse diagram of the concept lattice of the formal context (G, M, I) in Table 3 is shown in Fig. 8, where each formal concept (A, B) means that all the reduction methods in A are suitable for the types of consistent formal decision contexts in B.

5. Comparison of reductions

In this section, we compare the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions under I, II, III, and IV-consistent formal decision contexts since V-consistency is equivalent to III-consistency.

For our purpose, the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions are sometimes represented by Reductions I, II, III, IV, V and VI, respectively.

Definition 10. For $i, j \in \{I, II, III, IV, V, VI\}$ and $k \in \{I, II, III, IV, V\}$, let Reductions i and j be suitable for k-consistent formal decision contexts. If for any i-reduct E, there exists $F \subseteq E$ such that F is a j-reduct, then Reduction i is said to be coarser than Reduction j with respect to k-consistent formal decision contexts. We denote this relation by Reduction $i \leq_k$ Reduction j.

Moreover, if Reduction $i \leq_k$ Reduction j and Reduction $j \leq_k$ Reduction i, we say that Reduction i is equivalent to Reduction j with respect to k-consistent formal decision contexts, denoted by Reduction $i =_k$ Reduction j. If Reduction $i \leq_k$ Reduction j and Reduction $i \neq_k$ Reduction j, we say that Reduction i is strictly coarser than Reduction j with respect to k-consistent formal decision contexts, denoted by Reduction $i <_k$ Reduction j.

Proposition 8. *For $i, j \in \{I, II, III, IV, V, VI\}$ and $k \in \{I, II, III, IV, V\}$, let Reductions i and j be suitable for k-consistent formal decision contexts. Then, Reduction $i \leq_k$ Reduction j if and only if for any k-consistent formal decision context $K = (G, M, I, N, J)$, each i-reduct $E \subseteq M$ is a j-consistent set of K.*

Proof. Note that every j-consistent set E can lead to a j-reduct $F \subseteq E$ of K. Then, the proof is immediate from Definition 10. □

Now, we are ready to investigate the relationship among the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions under I, II, III, IV-consistent formal decision contexts.

5.1. Reduction comparison under I-consistent formal decision contexts

It can be seen from Fig. 8 that the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions are all suitable for I-consistent formal decision contexts. So, we need to compare all of them under I-consistent formal decision contexts.

Theorem 6. *For any I-consistent formal decision context $K=(G,M,I,N,J)$ with $E\subseteq M$, the following statements hold:*

(1) *if E is a I-reduct of K, then E is V, VI-reducts and II, III, IV-consistent sets of K;*
(2) *if E is a V-reduct of K, then E is a I-reduct of K;*
(3) *if E is a VI-reduct of K, then E is a I-reduct of K;*
(4) *if E is a III-reduct of K, then E is a II-consistent set of K.*

Proof. (1) Since E is a I-reduct of K, then we know from Section 3.1 that E is a minimal subset of M such that $K_E=(G,E,I_E,N,J)$ is I-consistent. Note that I-consistency is stronger than II, III, IV, V-consistencies according to Fig. 7. So, K_E is II, III, IV, V-consistent, which implies that E is II, III, IV, V-consistent sets of K.

Moreover, we prove that E is a V-reduct of K. Note that E has been proved to be a V-consistent set. So, if E is not a V-reduct of K, there exists $F\subset E$ such that F is a V-reduct of K. Since K is I-consistent, we conclude that every $(C,D)\in\underline{\mathfrak{B}}(G,N,J)$ with $C,D\neq\emptyset$ can and only can generate one non-redundant decision rule in $\mathcal{R}_F$. That is, F is a I-consistent set of K, which is in contradiction with the assumption that E is a I-reduct of K.

Besides, we prove that E is a VI-reduct of K. Note that every $(C,D)\in\underline{\mathfrak{B}}(G,N,J)$ with $C,D\neq\emptyset$ can and only can generate one non-redundant decision rule $B\to D$ in $\mathcal{R}_M$. Moreover, since E is a I-reduct of K, every $(C,D)\in\underline{\mathfrak{B}}(G,N,J)$ with $C,D\neq\emptyset$ can and only can generate one non-redundant decision rule $B_0\to D$ in $\mathcal{R}_E$. Then, we conclude that $B_0\to D$ implies $B\to D$ due to $\mathrm{Supp}(B_0\to D)=\mathrm{Supp}(B\to D)$. As a result, $\mathcal{R}_E^*\Rightarrow\mathcal{R}_M^*$. So, it can be known from Section 3.6 that E is a VI-consistent set of K.

If E is not a VI-reduct, there exists $F\subset E$ such that $\mathcal{R}_F^*\Rightarrow\mathcal{R}_M^*$. In other words, there exists $B_1\to D\in\mathcal{R}_F^*$ such that $B_1\to D$ implies $B\to D$ with $\mathrm{Supp}(B_1\to D)=\mathrm{Supp}(B\to D)$. Then, $B_1^{\prime_F}=C$. So, F is a I-consistent set of K, which is in contradiction with the assumption that E is a I-reduct of K.

(2) If E is a V-reduct of K, then every $(C,D)\in\underline{\mathfrak{B}}(G,N,J)$ with $C,D\neq\emptyset$ can lead to a non-redundant decision rule $B_0\to D$ in $\mathcal{R}_E$ with $\mathrm{Supp}(B_0\to D)=\mathrm{Supp}(B\to D)$. As a result, we get $B_0^{\prime_E}=C$, which implies that E is a I-consistent set of K. Moreover, it is easy to show that E is minimal with respect to I-consistency. Consequently, E is a I-reduct of K.

(3) If E is a VI-reduct of K, we have $\mathcal{R}_E^*\Rightarrow\mathcal{R}_M^*$. Since K is I-consistent, every $(C,D)\in\underline{\mathfrak{B}}(G,N,J)$ with $C,D\neq\emptyset$ can and only can generate one non-redundant decision rule $B\to D$ in $\mathcal{R}_M$. Based on $\mathcal{R}_E^*\Rightarrow\mathcal{R}_M^*$, there exists $B_0\to D\in\mathcal{R}_E^*$ such that $B_0\to D$ implies $B\to D$. Then, we can conclude $\mathrm{Supp}(B_0\to D)=\mathrm{Supp}(B\to D)$. That is, $B_0^{\prime_E}=C$. So, E is a I-consistent set of K. Moreover, it is obvious that E is minimal with respect to I-consistency. So, E is a I-reduct of K.

(4) Since E is a III-reduct of K, we conclude that $K_E=(G,E,I_E,N,J)$ is III-consistent. Based on Fig. 7, we know that II-consistency is weaker than III-consistency. So, K_E is also II-consistent. As a result, E is a II-consistent set of K. □

Based on Definition 10, Proposition 8 and Theorem 6, we have the following corollary.

Corollary 1. *For Reductions I, II, III, IV, V and VI, the following statements hold:*

(1) *Reduction $I=_I$ Reduction $V=_I$ Reduction $VI\leq_I$ Reduction $III\leq_I$ Reduction II;*
(2) *Reduction $I=_I$ Reduction $V=_I$ Reduction $VI\leq_I$ Reduction IV.*

It can be seen from Corollary 1 that Reductions I, V and VI are equivalent to one another with respect to I-consistent formal decision contexts. Thus, in what follows, we classify them into the same category and select Reduction I as a representative when this category is mentioned. In order to clarify the overall relationship among Reductions I, II, III, IV, V and VI under I-consistent formal decision contexts, we still need to verify whether or not the following statements are true:

(1) Reduction $II\leq_I$ Reduction I;
(2) Reduction $II\leq_I$ Reduction III;
(3) Reduction $II\leq_I$ Reduction IV;
(4) Reduction $III\leq_I$ Reduction I;
(5) Reduction $III\leq_I$ Reduction IV;
(6) Reduction $IV\leq_I$ Reduction I;
(7) Reduction $IV\leq_I$ Reduction II;
(8) Reduction $IV\leq_I$ Reduction III.

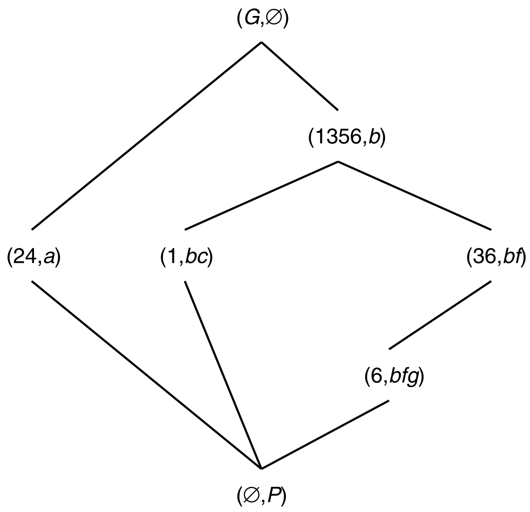

Fig. 9. $\underline{\mathcal{B}}(G, P, I_P)$.

In fact, none of the above statements holds. The following is a counterexample to confirm our assertion.

Example 2. Let $K = (G, M, I, N, J)$ be the formal decision context in Table 2. Then, it can be known from Figs. 1 and 2 that K is I-consistent.

First, let $P = \{a, b, c, f, g\}$. The Hasse diagram of the concept lattice of the subcontext (G, P, I_P) is shown in Fig. 9. Based on Figs. 2 and 9, P is a II-reduct of K. However,

- P is not a I-consistent set since we cannot find a concept from $\underline{\mathcal{B}}(G, P, I_P)$ such that its extent is equal to that of $(356, d_1d_3) \in \underline{\mathcal{B}}(G, N, J)$;
- P is not a III-consistent set since $(356, d_1d_3) \in \underline{\mathcal{B}}(G, N, J)$ cannot lead to a non-redundant decision rule in $\mathcal{R}_P$;
- P is not a IV-consistent set due to $5^{\prime_P \prime_P} = \{1, 3, 5, 6\} \nsubseteq \{3, 5, 6\} = 5^{\prime_N \prime_N}$.

Then, by Proposition 8, we conclude that

(1) Reduction $II \leq_I$ Reduction I,
(2) Reduction $II \leq_I$ Reduction III,
(3) Reduction $II \leq_I$ Reduction IV

are not true.

Second, let $Q = \{a, b, c, d, g\}$. Fig. 10 depicts the concept lattice of the subcontext (G, Q, I_Q). Then, according to Figs. 2 and 10, Q is a III-reduct of K. However,

- Q is not a I-consistent set since we cannot find a concept from $\underline{\mathcal{B}}(G, Q, I_Q)$ such that its extent is equal to that of $(36, d_1d_3d_4) \in \underline{\mathcal{B}}(G, N, J)$;
- Q is not a IV-consistent set due to $3^{\prime_Q \prime_Q} = \{3, 5, 6\} \nsubseteq \{3, 6\} = 3^{\prime_N \prime_N}$.

As a result,

(4) Reduction $III \leq_I$ Reduction I,
(5) Reduction $III \leq_I$ Reduction IV

are not satisfied.

Finally, take $R = \{a, c, d, f\}$. Fig. 11 depicts the concept lattice of the subcontext (G, R, I_R). Then, we can compute

$$
\begin{aligned}
&1^{\prime_R \prime_R} = \{1\} = 1^{\prime_N \prime_N},\\
&2^{\prime_R \prime_R} = \{2, 4\} = 2^{\prime_N \prime_N},\\
&3^{\prime_R \prime_R} = \{3, 6\} = 3^{\prime_N \prime_N},\\
&4^{\prime_R \prime_R} = \{2, 4\} = 4^{\prime_N \prime_N},\\
&5^{\prime_R \prime_R} = \{3, 5, 6\} = 5^{\prime_N \prime_N},\\
&6^{\prime_R \prime_R} = \{3, 6\} = 6^{\prime_N \prime_N}.
\end{aligned}
$$

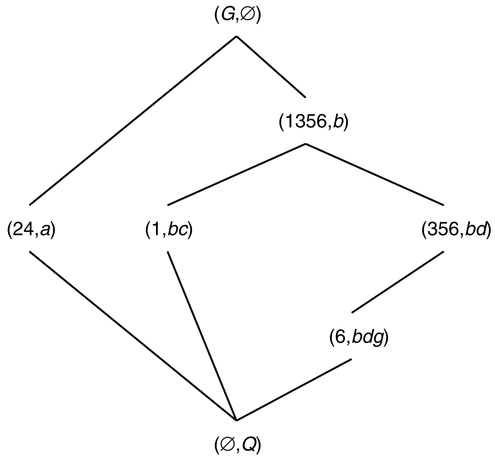

Fig. 10. $\underline{\mathfrak{B}}(G, Q, I_Q)$.

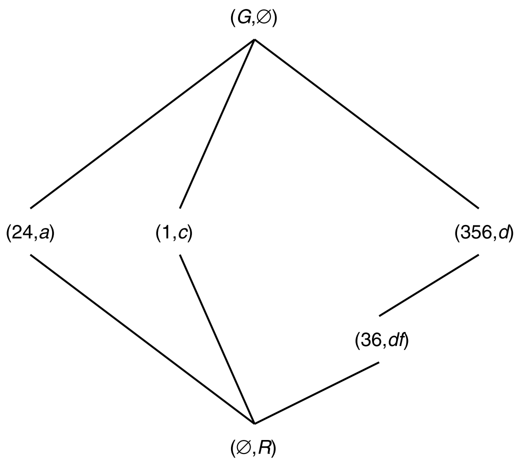

Fig. 11. $\underline{\mathfrak{B}}(G, R, I_R)$.

So, R is a IV-reduct of K. However, R is not I, II, III-consistent sets of K since the cardinality of $\underline{\mathfrak{B}}(G, N, J)$ is greater than that of $\underline{\mathfrak{B}}(G, R, I_R)$. As a result,

(6) Reduction $IV \leq_I$ Reduction I,
(7) Reduction $IV \leq_I$ Reduction II,
(8) Reduction $IV \leq_I$ Reduction III

are not true.

Based on the above discussion, Fig. 12 shows the relationship among the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions under I-consistent formal decision contexts in which the reduction becomes strictly coarser from bottom to top.

5.2. Reduction comparison under II-consistent formal decision contexts

In this section, we study the relationship among the reductions under II-consistent formal decision contexts. It can be known from Fig. 8 that only the Wei-2008b and Li-2011b reductions are suitable for II-consistent formal decision contexts. That is, we need to compare Wei-2008b and Li-2011b.

In what follows, we use a counterexample to show that

$$\text{Reduction } II \leq_{II} \text{Reduction } VI, \quad \text{Reduction } VI \leq_{II} \text{Reduction } II$$

do not hold. Here, it should be pointed out that Reductions II and VI represent Wei-2008b and Li-2011b, respectively.

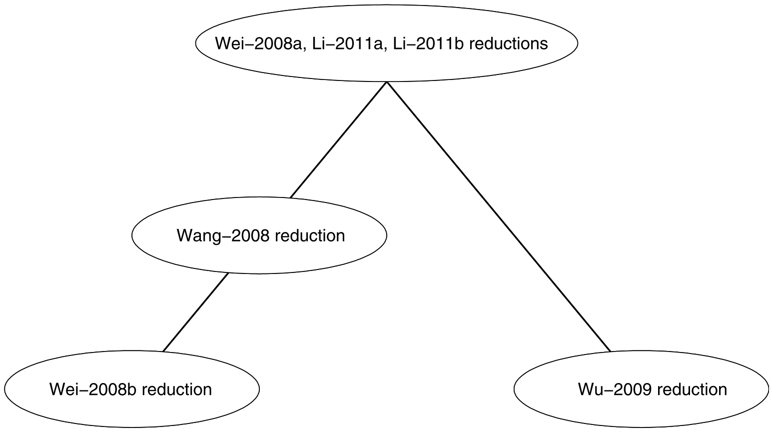

Fig. 12. The relationship among the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions under I-consistent formal decision contexts.

Table 4
A formal decision context $K = (G, M, I, N, J)$.

G	a	b	c	d	e	f	d_1	d_2	d_3	d_4
1	×						×			
2	×		×				×	×		
3		×		×				×	×	
4		×			×	×		×	×	×
5		×			×			×	×	×

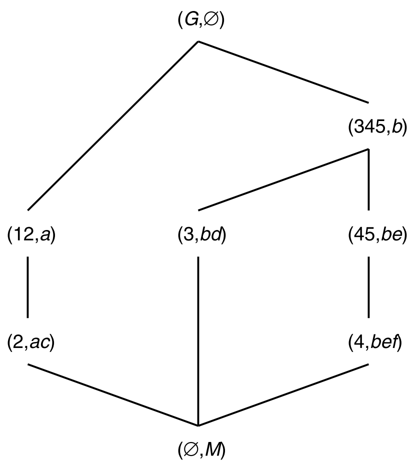

Fig. 13. $\underline{\mathfrak{B}}(G, M, I)$.

Example 3. Table 4 shows a formal decision context $K = (G, M, I, N, J)$, where $G = \{1, 2, 3, 4, 5\}$, $M = \{a, b, c, d, e, f\}$ and $N = \{d_1, d_2, d_3, d_4\}$.

The Hasse diagrams of the concept lattices of the formal contexts (G, M, I) and (G, N, J) are shown in Figs. 13 and 14, respectively. It can easily be seen from the figures that all the formal concepts $(C, D) \in \underline{\mathfrak{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate decision rules in $\mathcal{R}_M$ with their premises being pairwise different. Thus, K is II-consistent.

Let $L = \{a, c, d, e, f\}$. Fig. 15 depicts the Hasse diagram of the concept lattice of the subcontext (G, L, I_L). Then, it is easy to see that L is a II-reduct of K. However, L is not a VI-consistent set of K since the decision rule $b \to d_2d_3$ of $\mathcal{R}_M$ cannot be implied by any decision rule derived from (G, L, I_L, N, J). So, Reduction $II \leq_{II}$ Reduction VI does not hold.

Let $H = \{a, b, c, e\}$. Fig. 16 shows the Hasse diagram of the concept lattice of the subcontext (G, H, I_H). It can be known from Figs. 14 and 16 that H is a VI-reduct of K. However, H is not a II-consistent set of K since the cardinality of $\underline{\mathfrak{B}}(G, N, J)$ is greater than that of $\underline{\mathfrak{B}}(G, H, I_H)$. As a result, Reduction $VI \leq_{II}$ Reduction II is not true.

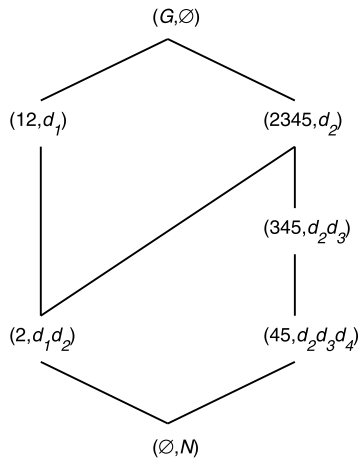

Fig. 14. $\underline{\mathcal{B}}(G, N, J)$.

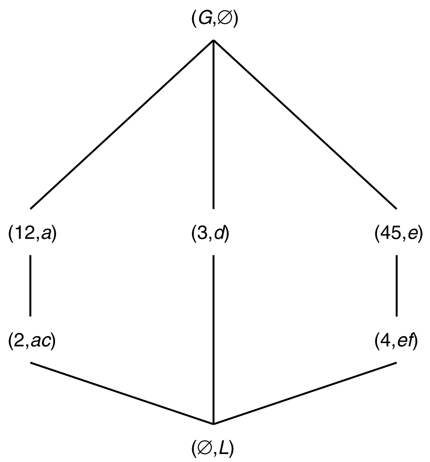

Fig. 15. $\underline{\mathcal{B}}(G, L, I_L)$.

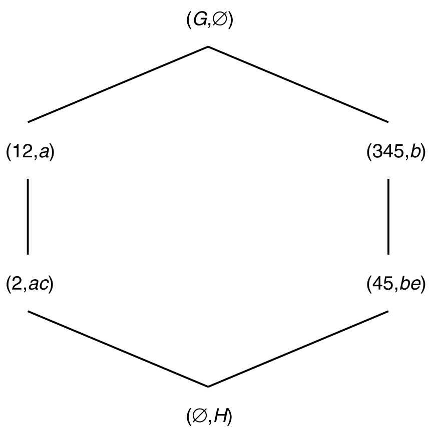

Fig. 16. $\underline{\mathcal{B}}(G, H, I_H)$.

Table 5
A formal decision context $K = (G, M, I, N, J)$.

G	a	b	c	d	e	f	d_1	d_2	d_3	d_4
1	×						×			
2					×			×		
3	×	×	×				×		×	×
4	×	×		×			×		×	
5						×		×		

5.3. Reduction comparison under III-consistent formal decision contexts

In this section, we continue to analyze the relationship among the reductions under III-consistent formal decision contexts. We know from Fig. 8 that the Wei-2008b, Wang-2008, Li-2011a and Li-2011b reductions are suitable for III-consistent formal decision contexts.

In what follows, we compare these four reductions. Note that Wei-2008b, Wang-2008, Li-2011a and Li-2011b are also represented by Reductions II, III, V and VI, respectively.

Theorem 7. *For any III-consistent formal decision context $K = (G, M, I, N, J)$ with $E \subseteq M$, the following statements hold:*

(1) *if E is a VI-reduct of K, then E is II, III, V-consistent sets of K;*
(2) *if E is a V-reduct of K, then E is II, III-consistent sets of K;*
(3) *if E is a III-reduct of K, then E is a II-consistent set of K.*

Proof. (1) Since E is a VI-reduct of K, we have $\mathcal{R}_E \Rightarrow \mathcal{R}_M$. Note that K is III-consistent. Then every $(C, D) \in \underline{\mathcal{B}}(G, N, J)$ with $C, D \neq \emptyset$ can generate a feedforward non-redundant decision rule $B \to D$ in $\mathcal{R}_M$. Moreover, among the feedforward non-redundant decision rules induced by (C, D), we choose such a decision rule $B_0 \to D$ that satisfies $\mathrm{Supp}(B_0 \to D) = \max_{B \to D \in \mathcal{R}_M} \{\mathrm{Supp}(B \to D)\}$. Based on $\mathcal{R}_E \Rightarrow \mathcal{R}_M$, there exists $B_1 \to D \in \mathcal{R}_E$ such that $B_1 \to D$ implies $B_0 \to D$ with $\mathrm{Supp}(B_1 \to D) = \mathrm{Supp}(B_0 \to D)$. As a result, $K_E = (G, E, I_E, N, J)$ is V-consistent. So, E is a V-consistent set of K.

Note that III-consistency is equivalent to V-consistency based on Fig. 12. Thus, E is also a III-consistent set of K. Besides, II-consistency is weaker than III-consistency, so E is a II-consistent set of K.

(2) Since E is a V-reduct of K, we know that $K_E = (G, E, I_E, N, J)$ is V-consistent. According to Fig. 12, III-consistency is equivalent to V-consistency. Then, $K_E = (G, E, I_E, N, J)$ is III-consistent. In other words, E is a III-consistent set of K. Moreover, II-consistency is weaker than III-consistency, which implies that K_E is II-consistent. That is, E is a II-consistent set of K.

(3) It is trivial since II-consistency is weaker than III-consistency. □

Based on Definition 10 and Theorem 7, we have the following corollary.

Corollary 2. *Reduction $VI \leq_{III}$ Reduction $V \leq_{III}$ Reduction $III \leq_{III}$ Reduction II.*

Furthermore, in order to obtain the overall relationship among Reductions II, III, V and VI under III-consistent formal decision contexts, we still need to check whether or not the following statements hold:

(1) Reduction $II \leq_{III}$ Reduction III;
(2) Reduction $II \leq_{III}$ Reduction V;
(3) Reduction $II \leq_{III}$ Reduction VI;
(4) Reduction $III \leq_{III}$ Reduction V;
(5) Reduction $III \leq_{III}$ Reduction VI;
(6) Reduction $V \leq_{III}$ Reduction VI.

In fact, none of the above statements holds. The following is a counterexample to confirm our assertion.

Example 4. Table 5 shows a formal decision context $K = (G, M, I, N, J)$, where $G = \{1, 2, 3, 4, 5\}$, $M = \{a, b, c, d, e, f\}$ and $N = \{d_1, d_2, d_3, d_4\}$.

The Hasse diagrams of the concept lattices of the formal contexts (G, M, I) and (G, N, J) are shown in Figs. 17 and 18, respectively. Then, it can be seen from the figures that K is III-consistent.

First, let $Z = \{b, c, d, e\}$. Fig. 19 depicts the Hasse diagram of the concept lattice of the subcontext (G, Z, I_Z). Then, Z is a II-reduct of K. However, it is not a III-consistent set of K since $(134, d_1) \in \underline{\mathcal{B}}(G, N, J)$ cannot generate a feedforward non-redundant decision rule in $\mathcal{R}_Z$. Note that III-consistency is equivalent to V-consistency. That is, Z is not a V-consistent

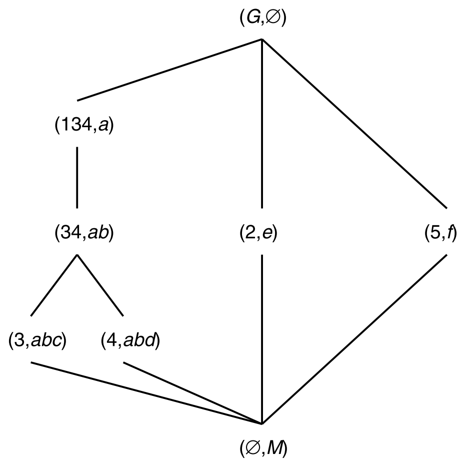

Fig. 17. $\underline{\mathcal{B}}(G, M, I)$.

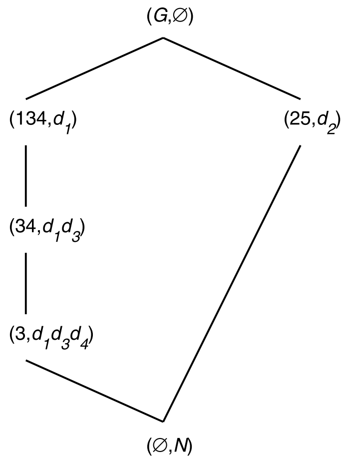

Fig. 18. $\underline{\mathcal{B}}(G, N, J)$.

set of K. Besides, $\mathcal{R}_Z \Rightarrow \mathcal{R}_M$ does not hold because $a \to d_1 \in \mathcal{R}_M$ cannot be implied by any decision rule derived from (G, Z, I_Z, N, J). In other words, Z is not a VI-consistent set of K. As a result,

(1) Reduction $II \leq_{III}$ Reduction III,
(2) Reduction $II \leq_{III}$ Reduction V,
(3) Reduction $II \leq_{III}$ Reduction VI

are not true.

Second, take $W = \{a, c, d, e\}$. Fig. 20 depicts the Hasse diagram of the concept lattice of the subcontext (G, W, I_W). Then, it is easy to prove that W is a III-reduct of K. However, W is not a V-consistent set of K since $(34, d_1d_3) \in \underline{\mathcal{B}}(G, N, J)$ cannot generate a non-redundant decision rule in $\mathcal{R}_W$ with its support being $\max\limits_{B \to D \in \mathcal{R}_M} \{\text{Supp}(B \to d_1d_3)\}$. Moreover, $\mathcal{R}_Z \Rightarrow \mathcal{R}_M$ does not hold because $ab \to d_1d_3 \in \mathcal{R}_M$ cannot be implied by any decision rule derived from (G, W, I_W, N, J). That is, W is not a VI-consistent set of K. So,

(4) Reduction $III \leq_{III}$ Reduction V,
(5) Reduction $III \leq_{III}$ Reduction VI

are not satisfied.

Finally, let $O = \{a, b, c, e\}$. We can verify that O is a V-reduct of K, but it is not a VI-consistent set of K since $f \to d_2 \in \mathcal{R}_M$ cannot be implied by any decision rule derived from (G, O, I_O, N, J). That is,

(6) Reduction $V \leq_{III}$ Reduction VI

is not true.

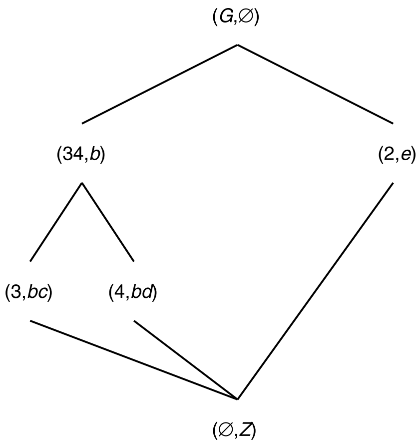

Fig. 19. $\underline{\mathfrak{B}}(G, Z, I_Z)$.

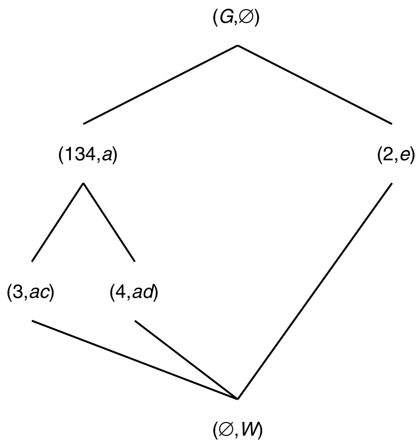

Fig. 20. $\underline{\mathfrak{B}}(G, W, I_W)$.

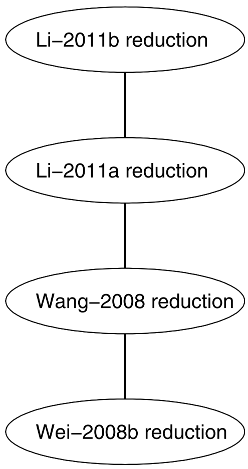

Fig. 21. The relationship among Reductions II, III, V and VI under III-consistent formal decision contexts.

Based on the above discussion, the relationship among the Wei-2008b, Wang-2008, Li-2011a and Li-2011b reductions under III-consistent formal decision contexts can be shown in Fig. 21 in which the reduction becomes strictly coarser from bottom to top.

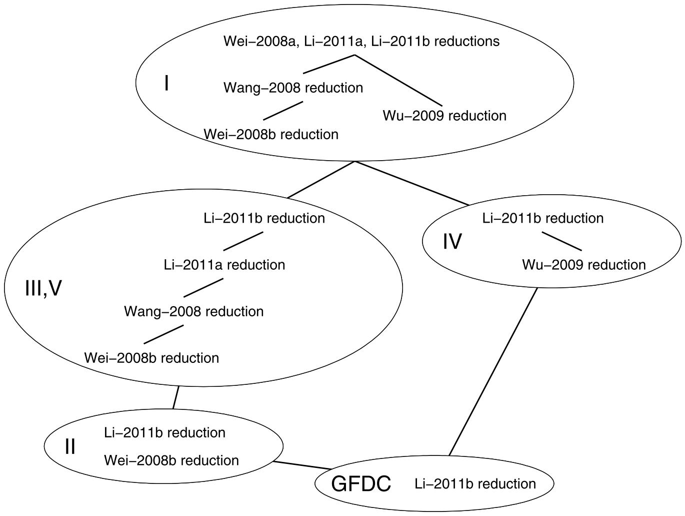

Fig. 22. The overall relationship among the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions under I, II, III, IV, V-consistent formal decision contexts.

5.4. Reduction comparison under IV-consistent formal decision contexts

According to Fig. 8, only the Wu-2009 and Li-2011b reductions are suitable for IV-consistent formal decision contexts. That is, we need to compare Wu-2009 and Li-2011b which are also represented by Reductions IV and VI, respectively.

Theorem 8. *For any IV-consistent formal decision context $K = (G, M, I, N, J)$, if $E \subseteq M$ is a VI-reduct of K, then it is a IV-consistent set of K.*

Proof. Since E is a VI-reduct of K, we have $\mathcal{R}_E \Rightarrow \mathcal{R}_M$. Note that K is IV-consistent. So, each $g \in G$ can generate a granular rule $g'^M \to g'^N$ in $\mathcal{R}_M$. Then there exists $B \to D \in \mathcal{R}_E$ such that $B \to D$ implies $g'^M \to g'^N$, which leads to $B \subseteq g'^M$ and $g'^N \subseteq D$. Moreover, $g \in B'^E \Rightarrow g'^{E'^E} \subseteq B'^E \subseteq D'^N \subseteq g'^{N'^N}$. As a result, E is a IV-consistent set of K. □

Based on Definition 10 and Theorem 8, we have the following corollary.

Corollary 3. *Reduction $VI \leq_{IV}$ Reduction IV.*

However, Reduction $IV \leq_{IV}$ Reduction VI does not hold. The following is a counterexample to confirm our assertion.

Example 5. Let $K = (G, M, I, N, J)$ be the formal decision context in Table 5. Then,

$$1'^{M'^M} = \{1, 3, 4\} = 1'^{N'^N},$$
$$2'^{M'^M} = \{2\} \subseteq \{2, 5\} = 2'^{N'^N},$$
$$3'^{M'^M} = \{3\} = 3'^{N'^N},$$
$$4'^{M'^M} = \{4\} \subseteq \{3, 4\} = 4'^{N'^N},$$
$$5'^{M'^M} = \{5\} \subseteq \{2, 5\} = 5'^{N'^N}.$$

That is, K is IV-consistent.

Let $E = \{a, c, d, e, f\}$. Then, for any $i \in \{1, 2, 3, 4, 5\}$, we have $i'^{E'^E} \subseteq i'^{N'^N}$, which means that E is a IV-consistent set. Moreover, we can prove that E is a IV-reduct of K. However, E is not a VI-consistent set of K since $ab \to d_1d_3 \in \mathcal{R}_M$ cannot be implied by any decision rule derived from (G, E, I_E, N, J). As a result,

$$\text{Reduction } IV \leq_{IV} \text{Reduction } VI$$

is not true.

In summary, Fig. 22 shows the overall relationship among the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reductions under I, II, III, IV, V-consistent formal decision contexts. Among the six ellipses, the order represents the strong and weak relations among the five consistencies. Here, GFDC denotes general formal decision contexts which are considered to be weaker than any kind of consistent formal decision contexts. What is more, within each ellipse, the order represents the coarse relation among the reductions under the certain consistent formal decision contexts.

The above findings may be used to evaluate the reduction of a formal decision context since the coarser a reduction method is, the less the attributes of a formal decision context can be reduced. Besides, the relationship among the reduction methods is beneficial to those who want to further discuss the reduction of a formal decision context since it can provide a reference for developing an appropriate kind of consistencies or preserving a certain type of rules.

Finally, it should be pointed out that the time complexity of the Wei-2008a, Wei-2008b, Wang-2008, Wu-2009, Li-2011a and Li-2011b reduction methods is all exponential since except Wu-2009, they are based on Wille's concept lattice. Although the Wu-2009 reduction method does not depend on any concept lattice, it involves the computation of all reducts which take exponential time. So, the detailed analysis of their time complexity is omitted here. We leave it to be discussed in our future work.

6. Final remarks

In this paper, we have given a rule-based review of the existing reduction methods in order to reveal the type of rules that each of them can preserve. Moreover, we have analyzed the relationship among their consistencies. In addition, a comparative study of their reductions has also been made. The obtained results can make researchers well understand the existing reduction methods and help users to select an appropriate reduction method to meet their requirements. More specifically, if users know the kind of consistencies that the input data satisfies and the type of rules that they want to preserve, then it will provide evidence to them in selecting an appropriate reduction method for the purpose of reducing the input data.

It should be noted that not only has attention been paid to knowledge reduction in classical formal decision contexts, but also there is a growing interest on this topic in generalized formal contexts or formal decision contexts [4,17,32], such as fuzzy formal contexts, fuzzy formal decision contexts, incomplete formal decision contexts, and real formal decision contexts. In our opinion, the results obtained in this paper may be beneficial to the further study of knowledge reduction in generalized formal decision contexts. This issue will be discussed in our forthcoming work.

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