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Fuzzy Sets and Systems 87 (1997) 385–387

FUZZY sets and systems

Short Communication

Some notes on the regularity of fuzzy measures on metric spaces

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Received January 1995; revised May 1996

Abstract

The purpose of this paper is to correct and modify some conclusions shown in a paper by Wu Congxin and Ha Minghu (*Fuzzy Sets and Systems* 66 (1994) 373–379) and to give some further results. © 1997 Elsevier Science B.V.

Keywords: Generalized measure theory; Fuzzy measures; Uniformly autocontinuity

In [1], Wu Congxin and Ha Minghu introduced the concept of regular fuzzy measure, gave some of its properties and showed Lusin's theorem. In this paper, we point out that the proof of Theorem 2.1 in [1] is incorrect, show a correct proof for it and give some further results generalizing the corresponding classical conclusions.

Let X be a metric space, B_X the class of all Borel sets on X and μ a fuzzy measure introduced by Sugeno [2]. (X, B_X, μ) is said to be a metric fuzzy measure space. In this paper, we always suppose that $\mu(X) < \infty$.

Definition 1 (Wang and Klir [3]). μ is said to be null-additive if $\mu(A \cup B) = \mu(A)$ whenever $A \in B_X$, $B \in B_X$, and $\mu(B) = 0$.

Definition 2 (Wang and Klir [3]). μ is said to be uniformly autocontinuous, if for arbitrary $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $\mu(A \cup B) \leqslant \mu(A) + \varepsilon$ whenever $A \in B_X, B \in B_X$, and $\mu(B) \leqslant \delta$.

It is obvious that the uniform autocontinuity implies the null-additivity.

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In [1], the authors have shown the following result.

Theorem 1 (Theorem 2.1 in [1]). *Let μ be any autocontinuous fuzzy measure on X, then μ is regular.*

We now point out that the proof of the theorem is not rigorous. Indeed, we cannot prove that $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{i=1}^{\infty} G_{n_i}^{(k)}$, so, $\bar{B}$ is not closed under countable unions, where $\bar{B}$ denotes the class of all Borel sets which are μ-regular in X. The proof of Theorem 1 is easy to modify as follows.

Proof of Theorem 1. It is enough to prove that $\bar{B}$ is closed under countable unions with the proof of the rest in Theorem 2.1 in [1]. Let $A_n \in \bar{B}, n = 1,2,\ldots,$ and $A = \bigcup_{n=1}^{\infty} A_n$. For any given $\varepsilon > 0$, since $A_1 \in \bar{B}$, there exist an open set G_1 and a closed set F_1 such that $F_1 \subset A_1 \subset G_1$ and $\mu(G_1 - F_1) < \varepsilon/2$. By using the autocontinuity of μ, we know that there exists δ_1 such that

$$\mu((G_1 - F_1) \cup B) < \mu(G_1 - F_1) + \frac{\varepsilon}{2^2},$$

whenever $B \in B_X$ and $\mu(B) < \delta_1$. From $A_2 \in \bar{B}$, we can choose an open set G_2 and a closed set F_2 such

PII S0165-0114(96)00161-3

that $F_2 \subset A_2 \subset G_2$ and $\mu(G_2 - F_2) < \delta_1$. Thus,

$$\mu((G_1 - F_1) \cup (G_2 - F_2)) < \left(\frac{1}{2} + \frac{1}{2^2}\right)\varepsilon.$$

Similarly, for set $(G_1 - F_1) \cup (G_2 - F_2)$ we can choose G_3 and F_3 such that $F_3 \subset A_3 \subset G_3$ and

$$\mu((G_1 - F_1) \cup (G_2 - F_2) \cup (G_3 - F_3)) < \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right)\varepsilon.$$

Continuing this procedure to infinite and using the continuity of μ, we obtain set sequences $\{G_n\}$ and $\{F_n\}$ satisfying $F_n \subset A_n \subset G_n$ for any $n = 1, 2, \ldots$, and $\mu(\bigcup_{n=1}^{\infty}(G_n - F_n)) \leqslant \varepsilon$.

Taking $G_\varepsilon = \bigcup_{n=1}^{\infty} G_n, F_\varepsilon = \bigcap_{n=1}^{\infty} F_n$, then $F_\varepsilon \subset A \subset G_\varepsilon$, G_ε is an open set and F_ε is closed set, as required. □

Although the proof of Theorem 2.1 in [1] is not strict, Corollary 2.1 in [1] is still correct.

Theorem 2 (Corollary 2.1 in [1]). *If μ is an autocontinuous fuzzy measure on X, then for every $A \in B_X$ and any $\varepsilon > 0$, there exists some closed F, such that $F \subset A$ and $\mu(A - F) < \varepsilon$.*

Proof. Let $\bar{B} = \{A \in B_X$: for each $\varepsilon > 0$, there exists a closed set $F \subset A, F \in B_X$ such that $\mu(A - F) < \varepsilon\}$. It is similar to Theorem 2.1 in [1]. □

By Theorem 2 we obtain Theorem 2.2 in [1].

In the following, we shall discuss further the properties of fuzzy measures on metric spaces.

First, we shall discuss the notion of spectrum or support of a fuzzy measure.

Theorem 3. *Let X be a separable metric space and μ a null-additive fuzzy measure of X. Then there exists a unique closed set C_μ such that (1) $\mu(C'_\mu) = 0$; (2) if D is any closed set such that $\mu(D') = 0$, then $C_\mu \subset D$. Moreover, C_μ is the set of all points $x \in X$ having the property that $\mu(U) > 0$ for each open set U containing x.*

Proof. Let $\Gamma = \{U\colon U$ is open, $\mu(U) = 0\}$. Since X is separable there are many countably open sets $U_1, U_2, \ldots$, such that $\bigcup_{n=1}^{\infty} U_n = \bigcup\{U\colon U \in \Gamma\}$. Let us denote this union by U_μ and set $C_\mu = X - U_\mu$. By the null-additivity of μ, we have $\mu(U_\mu) = \mu(\bigcup_{n=1}^{\infty} U_n) = 0$, i.e. $\mu(C'_\mu) = 0$. Further if D is any closed set with $\mu(D') = 0$, $X - D \in \Gamma$ and hence $X - D \subset U_\mu$, namely $C_\mu \subset D$. The uniqueness of C_μ is obvious. It remains to prove the last assertion. Notice that there exists an open set containing x for any $x \in C_\mu$ such that $\mu(U_\mu) = 0$, and $\mu(U)$ must be positive, if $x \in C_\mu$ and U is an open set containing x, we end the proof. □

Definition 3. The closed set C_μ in Theorem 3 is called the spectrum or support of μ.

Corollary 1. *Let X be any metric space and μ a null-additive fuzzy measure on X such that $\mu(X - E) = 0$, for some separable Borel set $E \subset X$. Then μ has a spectrum C_μ which is separable and $C_\mu \subset \bar{E}$.*

Second, we shall investigate a smaller class of fuzzy measures on metric spaces, the so-called tight fuzzy measures. Tight fuzzy measures have the property that they are determined by their values taken on compact sets.

Definition 4. A fuzzy measure μ on X is said to be tight if there exists a compact set $K_\varepsilon \subset X$ such that $\mu(X - K_\varepsilon) < \varepsilon$ for any given $\varepsilon > 0$.

Theorem 4. *Let μ be a tight uniformly autocontinuous fuzzy measure on X. Then μ has separable support and for any Borel set E and any $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset E$ such that $\mu(E - K_\varepsilon) < \varepsilon$.*

Proof. Let K_n be a compact set such that $\mu(X - K_n) < 1/n$. A compact set in a metric space is separable and, hence, $\bigcup_n K_n$ is separable. If $E_0 = \bigcup_n K_n$, then $\mu(E_0) = 0$ by the monotonicity of μ. The rest is similar to Theorem 2.3 in [1]. □

Theorem 5. *Let X be a separable metric space with the property that there exists a complete separable metric space $\tilde{X}$ such that X is contained in $\tilde{X}$ as a topological subset and X is a Borel subset of $\tilde{X}$. Then every uniformly autocontinuous fuzzy measure μ on X is tight. In particular, if X itself is a complete separable metric space, every uniformly autocontinuous fuzzy measure on X is tight.*

Proof. Let $X \subset \tilde{X}$, where $\tilde{X}$ is a complete separable metric space and X is a Borel set in $\tilde{X}$.

Given a fuzzy measure μ on B_X, we define $\tilde{\mu}$ on the class $B_{\tilde{X}}$ by setting $\tilde{\mu}(\tilde{A}) = \mu(\tilde{A} \cap X)$, $\tilde{A} \in B_{\tilde{X}}$. Since $X \in B_{\tilde{X}}$ $\tilde{\mu}(\tilde{X} - X) = 0$, we claim that it is enough to prove that $\tilde{\mu}$ is a tight measure on $\tilde{X}$. Indeed, since X is a Borel set in $\tilde{X}$, there will exist for each $\varepsilon > 0$, a set $K_\varepsilon \subset X$, compact in $\tilde{X}$, such that $\tilde{\mu}(X - K_\varepsilon) < \varepsilon$ (Theorem 4). K_ε is also compact in X since X is a topological subset of $\tilde{X}$. Further, $\mu(X - K_\varepsilon) = \tilde{\mu}(X - K_\varepsilon) < \varepsilon$. This implies that μ is tight. Thus, we may and do assume that X is itself a complete separable metric space. For the rest see Theorem 2.3 in [1]. □

Lastly, we shall discuss the notion of perfectness which is of some interest.

Definition 5. A fuzzy measure space (X, B, μ) is said to be perfect if for any B-measurable real valued function f and any set A on the real line such that $f^{-1}(A) \in B$, there are Borel sets A_1 and A_2 on the real line such that $A_1 \subset A \subset A_2$ and $\mu(f^{-1}(A_2 - A_1)) = 0$.

Theorem 6. *Let X be any metric space and μ a uniformly autocontinuous tight fuzzy measure on X. Then (X, B_X, μ) is a perfect fuzzy measure space.*

Proof. Let f be any real valued measurable function. It is sufficient to prove that for any $A \subset R^1$ (the real line) such that $f^{-1}(A) \in B_X$, there exists a Borel set $A_1 \subset A$ with $\mu(f^{-1}(A - A_1)) = 0$; A_2 can then be defined as a Borel set such that $A_2' \subset A'$ and $\mu(f^{-1}(A' - A_2')) = 0$. Suppose that $A \subset R^1$ is a set such that $E = f^{-1}(A) \in B_X$.

By Theorem 2.2, and Theorem 2.4 in [1] we can suppose that $\{C_n\}, n = 1, 2, \ldots,$ and $\{K_n\}, n = 1, 2, \ldots$ be two sequences of sets such that (1) $K_1 \subset K_2 \subset \cdots$, each K_n is compact, $f|K_n$ (f restricted to K_n) is continuous, and $\mu(X - K_n) \to 0$, (2) $C_1 \subset C_2 \subset \cdots \subset E$, each C_n is closed and $\mu(E - C_n) \to 0$.

If we write $\tilde{K}_n = K_n \cap C_n$, then $\tilde{K}_1 \subset \tilde{K}_2 \subset \cdots \subset E$, each $\tilde{K}_n$ is compact, $f|\tilde{K}_n$ is continuous and $\mu(E - \tilde{K}_n) \to 0$ as $n > \infty$. If $B_n = f(\tilde{K}_n)$ then B_n is a compact subset of the real line since $f|\tilde{K}_n$ is continuous and, hence, $A_1 = \bigcup_n B_n$ is a Borel set. Since, $f(\bigcup_n \tilde{K}_n) = A_1$ it follows that $f^{-1}(A_1) \supset \bigcup_n \tilde{K}_n$. Clearly, $A_1 \subset A$ and $f^{-1}(A_1) \subset f^{-1}(A) = E$. By the continuity of μ we obtain $\mu(E - \bigcup_n K_n) = \lim_n \mu(E - K_n) = 0$, so $\mu(E - f^{-1}(A_1)) = 0$. This completes the proof. □

Acknowledgements

The authors are greatly indebted to Prof. Dr. H.-J. Zimmermann and Prof. Dr. Wang Zhenyuan for their critical reading and many valuable suggestions which improved the manuscript.

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