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Fuzzy Sets and Systems 95 (1998) 77–81

FUZZY sets and systems

Fundamental convergence of sequences of measurable functions on fuzzy measure space[1]

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Received December 1995; revised September 1996

Abstract

The concepts of "fundamental almost everywhere" and "fundamental pseudo-almost everywhere" on fuzzy measure space are introduced, the relations among convergence of sequences of measurable functions are further discussed, the corresponding results on classical measure space are generalized and some of these results are improved in essence. © 1998 Elsevier Science B.V.

Keywords: Measure theory; Fuzzy measure

0. Introduction

The convergence of sequences of measurable functions on fuzzy measure space is one of the important contents of fuzzy measure theory. In [5–7], Wang gave some generalizations on the fuzzy measure space for Riesz's theorem, Egoroff's theorem and Lebesgue's theorem which are well known in classical measure theory. Now, we further discuss convergence of sequences of measurable functions by means of introducing the concepts of "fundamental almost everywhere" and "fundamental pseudo-almost everywhere" on fuzzy measure space. It is well known that the convergence in measure implies fundamental convergence in measure on classical measure space, but, since fuzzy measures lost additivity in general, the convergence in fuzzy measure and the fundamental convergence in fuzzy measure are not implied mutually on fuzzy measure space. In this paper, we shall point out that the depiction on fundamental convergence in fuzzy measure is just right the double asymptotic null-additivity introduced first by the authors. In addition, we further discuss the relations among the concepts of the other fundamental convergence of sequences of measurable functions on fuzzy measure space.

1. Preliminaries

Let X be a nonempty set, $\mathscr{L}$ a σ-algebra of subsets of X and $\mu: \mathscr{L} \to [0, \infty]$ a fuzzy measure, introduced first by Sugeno in [3] and generalized later by Ralescu and Adams in [2], i.e. with

[1]This work has been supported by the National Nature Science Foundation and a Special Foundation of Ph.D. in University of China.

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PII S0165-0114(96)00314-4

monotonicity and continuity from above and from below and $\mu(\emptyset) = 0$. We shall work throughout with the fixed fuzzy measure space $(X, \mathscr{L}, \mu)$ unless special explanation is given in the paper. For convenience, we recall some definitions and results.

Definition 1 (Wang Zhenyuan [5]). μ is said to be null-additive, if $\mu(E \cup F) = \mu(E)$ whenever $E \in \mathscr{L}$, $F \in \mathscr{L}$, $E \cap F = \emptyset$ and $\mu(F) = 0$.

Definition 2 (Wang Zhenyuan [5]). μ is said to be autocontinuous from above (resp., from below), if

$$\mu(A \cup B_n) \to \mu(A)$$

$$(\text{resp., } \mu(A - B_n) \to \mu(A))$$

whenever $A \in \mathscr{L}$, $B_n \in \mathscr{L}$, $n = 1, 2, \ldots$, and $\mu(B_n) \to 0$.

Definition 3 (Wang Zhenyuan [6]). Let $A \in \mathscr{L}$, and let P be a proposition with respect to points in A. If there exists $E \in \mathscr{L}$ with $\mu(E) = 0$ such that P is true on $A - E$, then we say "P is almost everywhere true on A". If there exists $F \in \mathscr{L}$ with $\mu(A - F) = \mu(A)$ such that P is true on $A - E$, then we say "P is pseudo-almost everywhere true on A".

From now on, the class of all nonnegative finite measurable functions is denoted by $\bar{F}$.

Definition 4 (Wang Zhenyuan [6]). Let $A \in \mathscr{L}$, $f \in \bar{F}$ and $\{f_n\} \subset \bar{F}$. If there exists $\{E_k\} \subset \mathscr{L}$ with $\lim_k \mu(E_k) = 0$ such that $\{f_n\}$ converges to f on $A - E_k$ uniformly for any fixed $k = 1, 2, \ldots$, then we say that $\{f_n\}$ converges to f on A almost uniformly, and denote it by $f_n \xrightarrow{\text{a.u.}} f$ (or $\lim_n f_n = f(\text{a.u.})$) on A. If there exists $\{F_k\} \subset \mathscr{L}$ with $\lim_k \mu(A - F_k) = \mu(A)$ such that $\{f_n\}$ converges to f on $A - F_k$ uniformly for any fixed $k = 1, 2, \ldots$, then we say that $\{f_n\}$ converges to f on A pseudo-almost uniformly, and denote it by $f_n \xrightarrow{\text{p.a.u.}} f$ (or $\lim_n f_n = f$ (p.a.u.)) on A.

Definition 5 (Wang Zhenyuan [6]). Let $A \in \mathscr{L}$, $f \in \bar{F}$ and $\{f_n\} \subset \bar{F}$. If

$$\lim_n \mu(\{x\colon |f_n(x) - f(x)| \geqslant \varepsilon\} \cap A) = 0$$

for any given $\varepsilon > 0$, then we say $\{f_n\}$ converges in μ to f on A, and denote it by $f_n \xrightarrow{\mu} f$ (or $\lim_n f_n = f(\mu)$) on A. If

$$\lim_n \mu(\{x\colon |f_n(x) - f(x)| < \varepsilon\} \cap A) = \mu(A)$$

for any given $\varepsilon > 0$, then we say $\{f_n\}$ converges pseudo-in μ to f on A, and denote it by $f_n \xrightarrow{\text{p.}\mu} f$ (or $\lim_n f_n = f$ (p.μ)) on A.

Definition 6 (Ha Minghu [1]). μ is said to have the property (S) (resp., (PS)), if for every sequence $\{E_n\} \subset \mathscr{L}$ with $\mu(E_n) \to 0$, there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_{n_i}\right) = 0$$

$$\left(\text{resp., } \mu\left(X - \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_{n_i}\right) = \mu(X)\right).$$

Proposition 1 (Wang Zhenyuan [6]). *If μ is autocontinuous from above (resp., from below), then μ has the property* (S) (*resp.*, (PS)).

Proposition 2 (Ha Minghu [1]). *Let $A \in \mathscr{L}$. μ has the property* (S) (*resp.*, (PS)) *if and only if for any $f \in \bar{F}$, $\{f_n\} \subset \bar{F}$, if $f_n \xrightarrow{\mu} f$ (resp. $f_n \xrightarrow{\text{p.}\mu} f$) on A, then there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \xrightarrow{\text{a.e}} f$ (resp., $f_{n_i} \xrightarrow{\text{p.a.e}} f$) on A.*

For the other concepts appearing in this paper see [4].

In the following, we give some definitions and a result.

Definition 7. μ is said to be double null-additive, if $\mu(E \cup F) = 0$ whenever $E \in \mathscr{L}$, $F \in \mathscr{L}$ and $\mu(E) = \mu(F) = 0$.

Definition 8. μ is said to be double pseudo-null-additive, if $\mu(B \cup C) = 0$, whenever $A \in \mathscr{L}$, $B \in \mathscr{L}$, $C \in \mathscr{L}$, $\mu(A) = \mu(C) = 0$ and $\mu(A - B) = 0$.

Definition 9. μ is said to be double asymptotic null-additive, if

$$\mu(A_n \cup B_m) \to 0 \quad (n \to \infty, m \to \infty)$$

whenever $\{A_n\} \subset \mathscr{L}$, $\{B_m\} \subset \mathscr{L}$, $\mu(A_n) \to 0$ and $\mu(B_m) \to 0$.

Definition 10. μ is said to be double asymptotic pseudo-null-additive, if

$$\mu(B_n \cup C_m) \to 0 \quad (n, m \to \infty)$$

whenever $\{A_n\} \subset \mathscr{L}$, $\{B_l\} \subset \mathscr{L}$, $\{C_m\} \subset \mathscr{L}$, $\mu(A_n) \to 0$, $\mu(C_m) \to 0$ and $\mu(A_n - B_l) \to 0$ $(n, l \to \infty)$.

Definition 11. Let $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$. If

$$\lim_{n,m} |f_n - f_m| = 0 \text{ (a.e.)}$$

(resp (a.u.), (μ), (p.a.e.), (p.a.u.), (p.μ)) on A,

then we say $\{f_n\}$ is fundamental (a.e.) (resp., (a.u.), (μ), (p.a.e.), (p.a.u.), (p.μ)) on A.

Proposition 3. *The double asymptotic null-additivity implies the double null-additivity.*

Proof. It is easy to prove by Definitions 7 and 9.

2. Principal results

By definitions and results above, we give the principal results of the paper as follows:

Proposition 4. *The autocontinuity from above implies the double asymptotic null-additivity.*

Proof. Suppose that the conclusion is not true, then there exist $\{A_n\} \subset \mathscr{L}$, $\{B_m\} \subset \mathscr{L}$, $\mu(A_n) \to 0$, $\mu(B_m) \to 0$ and ε_0 such that

$$\mu(A_n \cup B_m) \geqslant \varepsilon_0, \quad n, m = 1, 2, \ldots \tag{2.1}$$

By Proposition 1, there exist subsequence $\{A_{n_i}\} \subset \{A_n\}$ and $\{B_{m_k}\} \subset \{B_m\}$ such that

$$\mu\left(\limsup_i A_{n_i}\right) = 0, \qquad \mu\left(\limsup_k B_{m_k}\right) = 0.$$

Since autocontinuity implies null-additivity, by (2.1), we obtain

$$\begin{aligned} 0 &= \mu\left(\limsup_i A_{n_i}\right) \\ &= \mu\left(\limsup_i A_{n_i} \cup \limsup_k B_{m_k}\right) \\ &\geqslant \limsup_{i,k} \mu(A_{n_i} \cup B_{m_k}) > \varepsilon_0, \end{aligned}$$

a contradiction. This completes the proof of the proposition. □

The following example shows that the double asymptotic null-additivity is truly weaker than the autocontinuity from above.

Example 1. Let $X = \{0, 1, 2, \ldots\}$, $\mathscr{L} = P(X)$ (the power set of X), and let

$$\mu(E) = \begin{cases} \sum_{i \in E} \frac{1}{2^{i+1}}, & E \cap \{0\} = \emptyset, \\ 1, & E = \{0\}, \\ \infty, & \text{else.} \end{cases}$$

It is easy to prove that μ is a fuzzy measure. For arbitrary $\{A_n\} \subset \mathscr{L}$, $\{B_m\} \subset \mathscr{L}$, $\mu(A_n) \to 0$ and $\mu(B_m) \to 0$, by the construction of μ we obtain, $A_n \cap \{0\} = \emptyset$ and $B_m \cap \{0\} = \emptyset$ when n and m are large enough. As a result, $(A_n \cup B_m) \cap \{0\} = \emptyset$ when n and m are large enough, and

$$\begin{aligned} \mu(A_n \cup B_m) &\leqslant \sum_{i \in A_n} \frac{1}{2^{i+1}} + \sum_{i \in B_n} \frac{1}{2^{i+1}} \\ &= \mu(A_n) + \mu(B_m) \to 0 \quad (n, m \to \infty). \end{aligned}$$

It shows that μ is double asymptotic null-additive. But we can prove that μ is not autocontinuous from above. In fact, choose

$$A = \{0\}, \qquad B_n = \{n\}, \quad n = 1, 2, \ldots,$$

then $\mu(B_n) = 1/2^{n+1} \to 0$ $(n \to \infty)$ and $\mu(A \cup B_n) = \infty \nrightarrow \mu(A) = 1$.

It shows that μ is not autocontinuous from above.

Theorem 1. *μ is double asymptotic null-additive if and only if for every $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$, $\{f_n\}$ is fundamental in $[\mu]$ on A if there exists a $f \in \bar{F}$ such that $f_n \xrightarrow{\mu} f$ on A.*

Proof. We assume $A = X$ without loss of generality.

Necessity: By hypothesis we have

$$\lim_n \mu(\{x\colon |f_n(x) - f(x)| \geqslant \varepsilon/2\}) = 0,$$

$$\lim_m \mu(\{x\colon |f_m(x) - f(x)| \geqslant \varepsilon/2\}) = 0.$$

Notice, for arbitrary $\varepsilon > 0$, we have

$$\{x\colon |f_n(x) - f_m(x)| \geqslant \varepsilon\} \subset \{x\colon |f_n - f(x)| \geqslant \varepsilon/2\} \cup \{x\colon |f_m(x) - f(x)| \geqslant \varepsilon/2\}.$$

By the monotonicity and the double asymptotic null-additivity of μ, we have

$$\lim_{n,m} \mu(\{x\colon |f_n(x) - f_m(x)| \geqslant \varepsilon\}) = 0,$$

i.e. $\{f_n\}$ is fundamental in (μ) on A.

Sufficiency. Suppose there exist $\{A_n\} \subset \mathscr{L}$, $\{B_m\} \subset \mathscr{L}$ with $\mu(A_n) \to 0$ and $\mu(B_m) \to 0$ such that $\mu(A_n \cup B_m) \nrightarrow 0$. Let

$$f_{2n}(x) = I_{A_n}(x),$$
$$f_{2n-1}(x) = I_{B_n - A_n}(x), \quad n = 1, 2, \ldots, \quad f(x) \equiv 0,$$

where I is the membership function. It is obvious that $\{f_n\} \subset \bar{F}$, $f \in \bar{F}$, and for any ε, we have

$$\mu(\{x\colon |f_n(x) - f(x)| \geqslant \varepsilon\}) \leqslant \max\{\mu(A_n), \mu(B_n)\} \to 0$$

but

$$\mu(\{x\colon |f_{2n}(x) - f_{2n-1}(x)| \geqslant \varepsilon\}) = \mu(A_n \cup B_n) \nrightarrow 0,$$

i.e. μ has the double asymptotic null-additivity. This completes the proof of the theorem. □

Similar to Proposition 4 and Theorem 1, we obtain the following results:

Proposition 5. *The pseudo-autocontinuity from above implies the double asymptotic pseudo-null-additivity.*

Theorem 2. *μ is double asymptotic pseudo-null-additive if and only if for every $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$, $\{f_n\}$ is fundamental in $(p.\mu)$ on A if there exists a $f \in \bar{F}$ such that $f_n \xrightarrow{p.\mu} f$ on A.*

Referring to the proof of Theorem 6.8 in [7] and corresponding classical results, we obtain the following conclusions easily.

Theorem 3. *Let $A \in \mathscr{L}$, $\{f_n\} = \bar{F}$. If $\{f_n\}$ is fundamental (a.u.) (resp., (p.a.u.)) on A, then $\{f_n\}$ is fundamental (a.u.) (resp., (p.a.u.)) on A.*

Theorem 4. *Let $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$. $\{f_n\}$ converges to some $f \in \bar{F}$ on A almost everywhere (resp., pseudo-almost everywhere) if and only if $\{f_n\}$ is fundamental (a.u.) (resp., (p.a.u.)) on A.*

Theorem 5. *Let $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$. $\{f_n\}$ converges to some $f \in \bar{F}$ on A almost uniformly (resp., pseudo-almost uniformly) if and only if $\{f_n\}$ is fundamental (a.u.) (resp., (p.a.u.)) on A.*

Last, we give another generalization on fuzzy measure space for fundamental convergence.

Theorem 6. *Let $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$. If μ is double asymptotic null-additive (resp., pseudo-null-additive), then $\{f_n\}$ converges in μ to some $f \in \bar{F}$ on A if $\{f_n\}$ is fundamental in (μ) (resp., $(p.\mu)$) on A.*

Proof. We only prove the former, the latter is similar. We assume $A = X$ without loss of generality. Since $\{f_n\}$ is fundamental in (μ), by Proposition 2, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ is fundamental (a.e.). By Theorem 4, there exists $f \in \bar{F}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$ $(k \to \infty)$.

From Theorem 6.9 of [7], we know $f_{n_k} \xrightarrow{\mu} f$ $(k \to \infty)$. Since $\{f_n\}$ is fundamental in (μ), we have

$$\lim_{n,k} \mu(\{x\colon |f_n(x) - f_{n_k}(x)| \geqslant \varepsilon/2\}) = 0.$$

By $f_{n_k} \xrightarrow{\mu} f$ again, we have

$$\lim_k \mu(\{x\colon |f_{n_k}(x) - f(x)| \geqslant \varepsilon/2\}) = 0.$$

Notice that

$$\{x: |f_n(x) - f(x)| \geqslant \varepsilon\}$$
$$\subset \{x: |f_n(x) - f_{n_k}(x)| \geqslant \varepsilon/2\}$$
$$\cup \{x: |f_{n_k}(x) - f(x)| \geqslant \varepsilon/2\}.$$

By the monotonicity and double asymptotic null-additivity, we have

$$\lim_n \mu(x: |f_n(x) - f(x)| \geqslant \varepsilon\} = 0,$$

i.e. $f_n \xrightarrow{\mu} f$. This completes the proof of the theorem. □

Acknowledgements

The authors would like to thank Prof. H.-J. Zimmermann and referees for their critical reading of the manuscript and their valuable suggestions for improvements.

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