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Group theory-based optimization algorithm for solving knapsack problems \star

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ABSTRACT

This paper proposes a group theory-based optimization algorithm (GTOA) for knapsack problems, which draws algebraic group operations into the evolution process. The key parts of GTOA are that the feasible solution of the knapsack problem is considered as an element of the direct product of groups and that the evolution process is implemented by multiplication and inverse operations of the direct product of groups. Based on the algorithms for handling infeasible solutions, GTOA is used to solve knapsack problems such as the Set-union knapsack problem, the Discounted {0-1} knapsack problem, and the Bounded knapsack problem. GTOA is validated to be an efficient algorithm for solving knapsack problems. A comparison between GTOA and existing evolutionary algorithms such as genetic algorithm, binary particle swarm optimization, binary artificial bee colony, and their improved variations is conducted and the comparative results show that GTOA has a better performance than other algorithms. In addition, GTOA is not only an efficient algorithm for solving knapsack problems but is also the first paradigm that applies group theory to directly design an evolutionary algorithm.

1. Introduction

Knapsack problems (KPs) [1,2] are classical NP-complete problems and important combinatorial optimization problems as well. Modeling and solving KPs has great theoretical significance and practical value in many fields such as resource allocations, capital budgets, investment decisions, industrial loading, economy and finance, and information security [3,4]. The classic knapsack problem is the 0–1 knapsack problem (0-1KP), and there exist many extensions of 0-1KP such as the bounded knapsack problem (BKP),the unbounded knapsack problem (UKP), the multidimensional knapsack problem (MDKP), the multiple knapsack problem (MKP), the quadratic knapsack problem (QKP), the set-union knapsack problem (SUKP), the discounted {0-1} knapsack problem (D{0-1}KP), the randomized time-varying knapsack problem (RTVKP), the quadratic multiple knapsack problem (QMKP), and the multiple-choice multidimensional knapsack problem (MMKP) [3,5–10]. Though there are many members of the KPs-family, they can be divided into three categories according to their representation of feasible solutions.

The feasible solution of the first category of KPs can be represented as an *n*-dimensional 0–1 vector $Y = (y_1, y_2, \dots, y_n) \in \{0,1\}^n$, where *n* is the number of items in KPs. Obviously, 0-1KP, MDKP, QKP, D $\{0-1\}$ KP, SUKP, and RTVKP all belong to the first category of KPs. The second category of KPs use an *n*-dimensional integer vector $Y = (y_1, y_2, \dots, y_n) \in \{0, 1, \dots, m\}^n$ to represent a feasible solution, where *n* is the number of items in KPs, *m* is an integer and $m \ge 2$. MKP, QMKP, and D $\{0-1\}$ KP all belong to the second category of KPs. Note that D $\{0-1\}$ KP belongs to the first and second category of KPs simultaneously. The feasible solution of third category of KPs is an *n*-dimensional integer vector $Y = (y_1, y_2, \dots, y_n) \in \{0, 1, \dots, m_1\} \times \{0, 1, \dots, m_2\} \times \dots \times \{0, 1, \dots, m_n\}$, where *n* is the number of items in KPs, and m_1, m_2, \dots, m_n are *n*

 $\star\,$ Fully documented templates are available in the elsarticle package on CTAN.

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Input: $X = (x_1, x_2, ..., x_n)$, Mutation probability P_m . Output: The mutated $X = (x_1, x_2, ..., x_n)$. 1 for $i \leftarrow 1$ to n do 2 if $(rand_1 < P_m)$ then 3 if $(rand_2 \le 0.5 \land x_i \ne 0)$ then $x_i \leftarrow m_i + 1 - x_i$; 4 else $x_i \leftarrow rand([0, m_i] - \{x_i\})$; 5 end if 6 end for

Algorithm 1. IRMO.

Input: $X = (x_1, x_2, ..., x_n)$, Mutation probability P_m . Output: The mutated $X = (x_1, x_2, ..., x_n)$. 1 for $i \leftarrow 1$ to n do 2 if $(rand \le P_m)$ then $x_i \leftarrow 1 - x_i$; 3 end for

Algorithm 2. SMO.

Input: An instance of KPs; Parameters NP, MIT, m1, m2, ..., mn; Mutation probability Pm Output: An approximate solution (or optimal solution) B and f(B). 1 Generate initial population $P(0) = \{X_i(0) \mid 1 \le i \le NP\}$ randomly; 2 Compute $Fit(X_i(0))$, $1 \le i \le NP$, and $t \leftarrow 0$ 3 while (t < MIT) do for $i \leftarrow 1$ to NP do $Y \leftarrow X_{p_1}(t) \bigoplus (F(X_{p_2}(t) \bigoplus (-X_{p_3}(t))));$ $Y \leftarrow \text{IRMO}(Y, P_m); \text{ (or } Y \leftarrow \text{SMO}(Y, P_m);)$ 6 if $Fit(Y) > Fit(X_i(t))$ then $X_i(t+1) \leftarrow Y$; 8 else $X_i(t+1) \leftarrow X_i(t)$; end for 10 Generate P(t + 1), and $t \leftarrow t + 1$; 11 end while 12 Determine B in P(MIT); 13 return (B, f(B)). Algorithm 3. GTOA. Input: An instance of SUKP; Parameters NP, MIT, m_1, m_2, \ldots, m_n ; Mutation probability P_m .

```
Output: An approximate solution (or optimal solution) B and f(B).
1 Sorting p_i/T_i (i = 1, 2, ..., n) in descending order. Storing the subscripts of ordered p_i/T_i in array H[1...n]:
2 Generate initial population P(0) = \{X_i(0) \in \{0, 1\}^n \mid 1 \le i \le NP\} randomly, and t \leftarrow 0;
3 (X_i(t), f(X_i(t))) \leftarrow S - GROA(X_i(t), H[1 \dots n]), \ 1 \le i \le NP;
4 while(t < MIT) do
    for i \leftarrow 1 to NP do
         Y \leftarrow X_{p1}(t) \bigoplus (F(X_{p2}(t) \bigoplus (-X_{p3}(t))));
         Y \leftarrow SMO(Y, P_m)
         (Y, f(Y)) \leftarrow S - GROA(Y, H[1 \dots n]);
         if f(Y) > f(X_i(t)) then X_i(t+1) \leftarrow Y;
           else X_i(t+1) \leftarrow X_i(t);
10
      end for
11
      Generate P(t + 1), and t \leftarrow t + 1;
12
13 end while
14 Determine B in P(MIT);
15 return(B, f(B))
```

Algorithm 4. GTOA for SUKP.

positive integers in which at least two values are not equal. BKP, UKP, and MMKP all belong to the third category of KPs.

For solving KPs, there are many algorithms which can be mainly divided into two categories. One category consists of deterministic algorithms including dynamic programming, backtracking, and branch and bound [11,12]. The other category is non-deterministic algorithms which contains randomized algorithms, approximation algorithms, biological algorithms, and evolutionary algorithms [2,13–15]. Since there do not exist polynomial time deterministic algorithms to solve

KPs, non-deterministic algorithms are more suited to quickly and approximately solve the KPs in practical applications.

Evolutionary algorithms (EAs) [15,16] have been well recognized as algorithms with swarm intelligence, which are essentially stochastic approximations. Their main advantage is that they do not need to calculate the derivatives and gradients of the objective function, do not require the objective function to be continuous, and do have the inherent implicit parallelism and the strong ability to search for global optimization as well. The most classic EAs include the genetic algorithm (GA) [17], Input: A potential solution $X = (x_0, x_1, \dots, x_{n-1}) \in \{0, 1, 2, 3\}^n$ and array $H[1 \dots 3n]$. Output: A feasible solution $X = (x_0, x_1, \dots, x_{n-1})$ and its objective function value f(X). $1 \ R \leftarrow \sum_{i=0}^{n-1} \lceil \frac{x_i}{3} \rceil w_{3i+|x_i-1|}; \quad j \leftarrow 3n-1;$ 2 while (R > C) do //Repair phase $k \leftarrow \lfloor H[j]/3 \rfloor; r \leftarrow H[j](mod3);$ 3 if $(x_k = r + 1)$ then $x_k \leftarrow 0$; $R \leftarrow R - w_{H[j]}$; 4 5 $j \leftarrow j - 1;$ 6 end while 7 for $j \leftarrow 0$ to 3n - 1 do //Optimizing phase 8 $k \leftarrow \lfloor H[j]/3 \rfloor; r \leftarrow H[j](mod3);$ if $(x_k = 0) \land (R + w_{H[j]} \le C)$ then 10 $R \leftarrow R + w_{H[j]}; x_k \leftarrow r + 1;$ 11 end if

12 end for 13 return(X, f(X)).

Algorithm 5. D-GROA.

particle swarm optimization (PSO) [18], differential evolution (DE) [19], harmony search algorithm (HS) [20], artificial fish swarm (AFS) [21], ant colony optimization (ACO) [22], and artificial bee colony (ABC) [23]. In recent years, EAs have had many successful applications in solving optimization problems, such as numerical optimization problems [24,25], knapsack problems [26], satisfiability problems [27], and set cover problems [28], which attract the great attention of scholars in many fields. Currently, by simulating behaviors of biological colonies in nature or learning from some phenomena of social activities, researchers have put forward many new EAs consecutively, such as the firefly algorithm (FFA) [29], the sine cosine algorithm (SCA) [30], the symbiotic organisms search (SOS) [31], the grey wolf optimizer (GWO) [32], and teaching-learning-based optimization (TLBO) [33]. Although nature provides us with endless inspiration, it is not perfect to design EAs solely based on bionic thought. In addition, the performance of many newly proposed EAs is not better than the classic ones, which accords with the No Free Lunch Theorem [34]. It is well known that, except for GA and ACO, almost existing EAs were initially put forward to solve numerical optimization problems and they generally could not be applied to directly solve combinatorial optimization problems. The methods proposed in the literature [35-37] that modify the evolution operator to fit to solve problems and in the literature [38,39] that map a real vector to a potential solution of problem are suitable only for the first category of KPs, not for the second and third ones.

Recently, the literatures [40,41] has used permutation group to improve the existing evolutionary algorithms, which are given an abstract algebraic differential mutation of DE and a vector operations of PSO, respectively, and used them to provide approaches to solving permutation problems. In fact, group theory not only can be used to improve the existing evolutionary algorithms, but also can be used to directly design a

new evolutionary algorithm. In this paper, we will take the knapsack problems as an example to illustrate how to design a new evolutionary algorithm by using the direct product of groups which is generated by the additive group of Z/nZ, the integers modulo n, where Z is the integer set. From the point of view of algebra, we can see that no matter which category the KPs belong to, each dimensional component of its feasible solutions can be considered as an element in the additive group of Z/nZ which can be simplified as $Z_n = \{0, 1, \dots, n-1\}$, where *n* is a positive integer and $n \ge 2$. Based on the abovementioned observation, an algebraic methodology for designing EAs is presented, and a Group Theory-based Optimization Algorithm (GTOA) is proposed by applying the direct product of groups. GTOA has a universal property and is suitable for solving KPs. The computations of SUKP, D{0-1}KP, and BKP illustrate that GTOA is not only easy to implement but also highly efficient. Moreover, GTOA is also a successful example of algorithm design based on group theory.

2. The proposed group theory-based optimization algorithm

2.1. Group and direct product of groups

The following is a brief introduction to the basic concepts and terminologies of the group and direct product of groups. More details can be found from literatures [42,43].

Let *G* be a nonempty set, and let * be a binary operation on *G*, if

(1) $\forall a, b \in G, a^*b \in G;$

(2) $\forall a,b,c \in G, (a^*b)^*c = a^*(b^*c);$

(3) $\forall a \in G, \exists e \in G, a^*e = e^*a = a;$

(4) $\forall a \in G, \exists b \in G, a^*b = b^*a = e$, then (G, *) is a group.

Henceforth, we shall write ab instead of a^*b and G instead of (G, *). We call e the identity of G and, if ab = ba = e, then we call b the inverse of a, denoted by a^{-1} . It is clear that the inverse of e is itself.

Let *G* be a group, $a \in G$, and let *i* be an integer. Then, the *i*th power a^i of *a* is defined as following:

$$a^{i} = \begin{cases} a^{i-1}a, & \text{if } i > 0; \\ e, & \text{if } i = 0; \\ \left(a^{-1}\right)^{-i}, & \text{if } i < 0. \end{cases}$$
(1)

Let $Z_n = \{0, 1, \dots, n-1\}$, *n* is a positive integer and $n \ge 2$. We define a binary operation \bigoplus on Z_n as follows:

$$\forall a, b \in \mathbb{Z}_n, a \bigoplus b = (a+b) (modn).$$
⁽²⁾

```
Input: An instance of D{0-1}KP; Parameters NP, MIT, m_1, m_2, \ldots, m_n; Mutation probability P_m
   Output: An approximate solution (or optimal solution) B and f(B).
   1 Sort p_i/w_i (j = 0, 1, ..., 3n - 1) in descending order. Store the subscripts of ordered p_i/w_i in array H[0...3n - 1];
   2 Generating initial population P(0) = \{X_i(0) \in \{0, 1, 2, 3\}^n \mid 1 \le i \le NP\} randomly, t \leftarrow 0;
   3 (X_i(t), f(X_i(t))) \leftarrow \boldsymbol{D} - \boldsymbol{GROA}(X_i(t), H[0 \dots 3n-1]), 1 \le i \le NP;
   4 while(t < MIT) do
   5
         for i \leftarrow 1 to NP do
   6
              Y \leftarrow X_{p1}(t) \bigoplus (F(X_{p2}(t) \bigoplus (-X_{p3}(t))));
  7
              Y \leftarrow IRMO(Y, P_m);
              (Y, f(Y)) \leftarrow \mathbf{D} - \mathbf{GROA}(Y, H[0 \dots 3n-1]);
   8
             if f(Y) > f(X_i(t)) then X_i(t+1) \leftarrow Y
   10
              else X_i(t+1) \leftarrow X_i(t);
11
          end for
           Generate P(t + 1), and t \leftarrow t + 1;
   13 end while
   14 Determine B in P(MIT);
   15 return(B, f(B)).
```

Algorithm 6. GTOA for D{0-1}KP.









= 0.005, 0.006, 0.007, 0.008, 0.009, 0.010, and 0.011

where + is a common addition operator, x(modn) denotes the remainder when x is divided by n.

It is easy to see that Z_n is a group for operation \bigoplus , and its identity is 0. We denote -a as the inverse of a in Z_n . Then, -0=0, and, if $a \neq 0$ then -a = n - a.

If G_1, G_2, \dots, G_k are *k* groups, then their direct product, denoted by $G_1 \times G_2 \times \dots \times G_k$, is the group with elements that are all ordered *k* tuple (a_1, a_2, \dots, a_k) , where $a_i \in G_i (i = 1, 2, \dots, k)$, and with operation

$$(a_1, \cdots, a_k) (b_1, \cdots, b_k) = (a_1 b_1, \cdots, a_k b_k).$$
⁽³⁾







Fig. 4. Gap fitting curve of the 1st category of SUKP instances.



Fig. 5. Gap fitting curve of the 2nd category of SUKP instances.

It is easy to check that $G_1 \times G_2 \times \cdots \times G_k$ is a group: the identity is (e_1, e_2, \dots, e_k) where e_i is the identity of G_i $(i = 1, 2, \dots, k)$ and the inverse $(a_1, a_2, \dots, a_k)^{-1}$ is $(a_1^{-1}, a_2^{-1}, \dots, a_k^{-1})$.

Let $Z_{n_i} = \{0, 1, \dots, n_i - 1\}$, where n_i is a positive integer and $n_i \ge 2$, $i = 1, 2, \dots, k$. Then, $Z_{n_1} \times Z_{n_2} \times \dots \times Z_{n_k}$ is a direct product of groups, denoted by $Z[n_1, n_2, \dots, n_k]$. It is clear that $(0, 0, \dots, 0)$ is the identity, and the inverse of (a_1, a_2, \dots, a_k) is $- (a_1, a_2, \dots, a_k)$, and

$$-(a_1, a_2, \cdots, a_k) = (-a_1, -a_2, \cdots, -a_k).$$
(4)



Fig. 6. Gap fitting curve of the 3rd category of SUKP instances.



Fig. 7. StD histogram of the 1st category of SUKP instances.



Fig. 8. StD histogram of the 2nd category of SUKP instances.

where $-a_i$ is the inverse of a_i in Z_{n_i} . $\forall (a_1, a_2, \dots, a_k)$, $(b_1, b_2, \dots, b_k) \in \mathbb{Z}[n_1, n_2, \dots, n_k]$, we have

$$(a_1, \dots, a_k) \bigoplus (b_1, \dots, b_k) = (a_1 \bigoplus b_1, \dots, a_k \bigoplus b_k).$$
(5)

The *i*th power of (a_1, a_2, \dots, a_k) is $i(a_1, a_2, \dots, a_k)$, where *i* is an integer.

It is easy to see that $Z[2,2,..,2] = \{0,1\}^n$, $Z[m + 1,m + 1,...,m + 1] = \{0,1,...,m\}^n$ and $Z[m_1 + 1,m_2 + 1,...,m_n + 1] =$ $\{0,1,...,m_1\} \times \{0,1,...,m_2\} \times \dots \times \{0,1,!,m_n\}$



Fig. 9. StD histogram of the 3rd category of SUKP instances.



Fig. 10. Gap fitting curve of UDKP1-10.



Fig. 11. Gap fitting curve of WDKP1-10.



It is easy to see that the feasible solutions of the first category of KPs are the elements in the direct product of groups Z[2,2,...,2]; those of the second category are the elements in Z[m + 1,m + 1,...,m + 1]; and those of the third category are the elements in $Z[m_1 + 1,m_2 + 1,...,m_n + 1]$. We notice that Z[2,2,...,2] and Z[m + 1,m + 1,...,m + 1] are special cases of $Z[m_1 + 1,m_2 + 1,...,m_n + 1]$, respectively. Hence, the basic principle and the pseudo-code of GTOA will be described by applying the direct product $Z[m_1 + 1,m_2 + 1,...,m_n + 1]$.



Suppose that $Y = (y_1, y_2, \dots, y_n)$, $V = (v_1, v_2, \dots, v_n)$, $W = (w_1, w_2, \dots, w_n)$ are three different elements randomly selected from $Z[m_1 + 1, m_2 + 1, \dots, m_n + 1]$. A new element $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ is generated by manipulating a group operation on *Y*, *V*, *W* according to the following equation:

$$X = Y \bigoplus \left(F \left(V \bigoplus (-W) \right) \right).$$
(6)

where $x_j = y_j \bigoplus [f_j(v_j \bigoplus (m_j + 1 - w_j))], j = 1, 2, \dots, n; F = (f_1, f_2, \dots, f_n)$ is an *n*-dimensional random vector in Knowledge-Based Systems xxx (2018) xxx-xxx





Fig. 16. StD histogram of SDKP1-10.





 $\{-1,0,1\}^n$, which is called the Combinatorial Factor Vector. Since formula (6) is similar to the representation of a straight-line equation, we call it the Random Linear Combination Operator (RLCO).

It is worth to note that RLCO uses three different random elements to produce a new element on the space $Z[m_1 + 1, m_2 + 1, \dots, m_n + 1]$, which indicates it is not only a random operation but has the characteristic of learning from other elements. Therefore, RLCO is a global random search operator and exhibits the global exploration ability by learning from others. However, it is not enough for an evolutionary al-

Table 1	L
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Comparison among GTOA, GA, BPSO and BABC for the 1st category of SUKP.

Index	Instance	CBEST	Algorithm	Best	Mean	Worst	StD	Time
1	sukp100_85	13,283	GA	13044	12956.4	12,596	130.66	0.112
	_0.10_0.75		BPSO	13082	12979.2	12,497	147.28	0.199
			BABC	13251	13028.5	12763	92.63	0.210
			GTOA	13251	13025.4	12763	59.70	0.134
2	sukp100_85	12,479	GA	12066	11546.0	11,296	214.94	0.119
	0.15 0.85		BPSO	12238	12089.0	11,576	128.99	0.225
			BABC	12238	12155.0	12066	53.29	0.223
			GTOA	12274	12029.3	11,483	127.25	0.150
3	sukp200_185	13,405	GA	13064	12492.5	12,596	320.03	1.013
	0.10_0.75		BPSO	13241	12831.6	11,687	434.69	1.640
			BABC	13241	13064.4	12,808	99.57	1.562
			GTOA	13405	13196.9	12932	127.43	1.160
4	sukp200_185	14,215	GA	13671	12802.9	12,332	291.66	1.133
	0.15 0.85		BPSO	14044	13380.7	12,782	332.27	1.737
			BABC	13829	13359.2	12881	234.99	1.729
			GTOA	14215	13285.9	12,763	276.35	1.277
5	sukp300_285	11,413	GA	10553	9980.9	9640	142.97	3.608
	0.10_0.75		BPSO	10869	10371.9	9708	240.20	5.759
			BABC	10428	9994.8	9661	154.03	5.281
			GTOA	11407	10674.5	10032	599.55	3.957
6	sukp300 285	12,245	GA	11016	10349.8	9906	215.13	3.899
	0.15 0.85		BPSO	12245	11034.1	10,056	581.86	5.812
			BABC	12012	10902.9	9929	449.45	5.673
			GTOA	12245	11533.4	10382	466.27	4.144
7	sukp400 385	11,435	GA	10083	9641.9	9370	168.94	9.779
	0.10 0.75		BPSO	11230	10580.0	9915	329.43	13.375
			BABC	10766	10065.2	9515	241.45	12.976
			GTOA	11435	10830.3	10051	353.33	9.165
8	sukp400_385	10,397	GA	9831	9326.8	8980	192.20	9.978
	0.15 0.85		BPSO	9990	9500.4	9081	278.67	15.076
			BABC	9649	9136.0	8894	151.90	13.359
			GTOA	10397	9894.6	9142	290.02	9.973
9	sukp500 485	11,716	GA	11031	10567.9	10,288	123.15	18.198
	0.10 0.75		BPSO	11473	10839.5	10,322	311.32	26.049
			BABC	10784	10452.2	10,257	114.35	25.372
			GTOA	11716	11171.4	10591	299.00	18.476
10	sukp500_485	9892	GA	9472	8692.7	8400	180.12	19.720
	0.15_0.85		BPSO	9446	9012.9	8627	197.57	29.790
			BABC	9090	8857.9	8651	94.55	26.874
			GTOA	9860	9262.0	8742	153.89	20.053

gorithm to only have the global exploration ability. An excellent evolutionary algorithm should make a trade-off between exploration and exploitation, which are both indispensable factors for an evolutionary algorithm to work efficiently. In the following, we propose a local search operator over the space $Z[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ based on the inverse operation, which is named the Inversion and Random Mutation Operator (IRMO).

Let $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ and $m_i \ge 1$ $(i = 1, 2, \dots, n)$, P_m be the mutation probability of IRMO. Then, the pseudo-code of IRMO is described as below.

In IRMO, $rand_1$ and $rand_2$ are two random numbers in the interval (0, 1); the mutation probability P_m satisfies $0 < P_m \le 0.5$; $rand([0,m_i] - \{x_i\})$ represents a random integer that is not equal to x_i in $[0, m_i]$. It is obvious that the probabilities of obtaining the inverse mutation (Step3) and the random mutation (Step4) are equal. The time complexity of IRMO is O(n).

In addition, in group $Z_2 = \{0,1\}$ under modulo 2, since -0 = 0 and -1 = 1, the IRMO does not work for 0–1 vectors in the search space Z[2,2,...,2]. Drawing lessons from the mutation operator of GA [17], a local search operator suitable for the space Z[2,2,...,2] is proposed, we name it the Switch Mutation Operator (SMO).

Let $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}[2, 2, \dots, 2]$ and P_m be the mutation probability of SMO. Then, the pseudo-code of SMO is described as follow:

In SMO, *rand* is a random number in (0,1), and the mutation probability P_m satisfies $0 < P_m \le 0.5$. It is easy to see that the time complexity of SMO is O(n).

Let $P(t) = \{X_i(t) | 1 \le i \le NP\}$ be the *t*-th generation population of where GTOA. $X_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t)) \in \mathbb{Z}[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ is the *i*-th individual, NP is the size of the population, t is an integer and $t \ge 0$. $Fit(X_i(t))$ denotes the fitness of the individual $X_i(t)$ and its computarelies on the tion generally objective function. Let $B = (b_1, b_2, \dots, b_n) \in \mathbb{Z}[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ be the best individual in P(MIT), MIT be the number of maximum iterations, and $Y = (y_1, y_2, \dots, y_n) \in \mathbb{Z}[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ be an *n*-dimensional vector. For the KPs with the objective function Maxf(X), the

pseudo-code of GTOA is described as follows: 5 of GTOA, individual In Step а temporary $Y = (y_1, y_2, \dots, y_n) \in \mathbb{Z}[m_1 + 1, m_2 + 1, \dots, m_n + 1]$ is generated by the RLCO operator, where $X_{p_1}(t)$, $X_{p_2}(t)$ and $X_{p_3}(t)$ are three different individuals randomly selected from the t-th generation population P(t). The individual Y is mutated by SMO in Step 6 when solving the first category of KPs and by IRMO when solving those in the second and third categories. Noting that both MIT and NP are multiples of n, the time complexity Comparison among GTOA, GA, BPSO and BABC for the 2nd category of SUKP.

*								
Index	Instance	CBEST	Algorithm	Best	Mean	Worst	StD	Time
1	sukp100_100	14,044	GA	14044	13806.0	13587	144.91	0.129
	_0.10_0.75		BPSO	14044	13846.1	13664	62.21	0.307
			BABC	13860	13734.9	13573	70.76	0.213
			GTOA	14044	13792.5	13561	90.57	0.161
2	sukp100_100	13,508	GA	13145	12234.8	11582	388.66	0.143
	_0.15_0.85		BPSO	13508	13428.9	13104	115.87	0.270
			BABC	13508	13352.4	12837	155.14	0.244
			GTOA	13508	13220.5	11988	296.61	0.175
3	sukp200_200	12,350	GA	11656	10888.7	10337	237.85	1.106
	_0.10_0.75		BPSO	12019	11344.8	10641	330.30	2.221
			BABC	11846	11194.3	10581	249.58	1.633
			GTOA	12350	11983.4	10903	326.08	1.302
4	sukp200_200	12,317	GA	11792	10827.5	10304	334.43	1.183
	_0.15_0.85		BPSO	11821	11357.2	10607	381.88	1.922
			BABC	11521	10945.0	10436	255.14	1.819
			GTOA	12317	11572.8	10957	242.36	1.410
5	sukp300_300	12,695	GA	12055	11755.1	11169	144.45	3.789
	_0.10_0.75		BPSO	12644	12187.6	11807	180.18	6.166
			BABC	12186	11945.8	11724	127.80	5.315
			GTOA	12695	12464.1	11968	231.68	4.190
6	sukp300_300	11,425	GA	10666	10099.2	9549	337.42	4.106
	_0.15_0.85		BPSO	11007	10409.4	9463	304.99	6.707
			BABC	10382	9859.7	9476	177.02	6.019
			GTOA	11425	10513.9	9477	355.88	4.398
7	sukp400_400	11,490	GA	10570	10112.4	9786	157.89	9.187
	_0.10_0.75		BPSO	11310	10600.5	10022	271.05	13.419
			BABC	10626	10101.1	9756	196.99	12.805
			GTOA	11450	10951.9	10360	264.34	9.695
8	sukp400_400	10,915	GA	9235	8793.8	8501	169.52	9.830
	_0.15_0.85		BPSO	10404	9383.6	8597	411.48	15.103
			BABC	9541	9033.0	8553	194.18	12.953
			GTOA	10915	9834.43	9145	312.87	10.429
9	sukp500_500	10,960	GA	10460	10185.4	9919	114.19	20.717
	_0.10_0.75		BPSO	10888	10522.4	10139	166.42	28.253
			BABC	10755	10328.5	10139	91.62	27.735
			GTOA	10960	10626.1	10048	152.96	18.592
10	sukp500_500	10,194	GA	9496	8882.9	8577	158.21	20.379
	_0.15_0.85		BPSO	9840	9447.9	8731	202.16	32.389
			BABC	9318	9180.8	8833	84.91	27.813
			GTOA	10194	9754.7	9044	231.77	20.291

of GTOA is $O(MIT^*n^*NP) = O(n^3)$, which indicates that GTOA is a stochastic approximation algorithm.

3. Solving Knapsack problems by GTOA

To illustrate the effectiveness and wide application range of GTOA, we select three typical knapsack problems, i.e., SUKP, D{0-1}KP, and BKP, and use GTOA to solve them. Since these three problems are constrained optimization problems, their infeasible solutions will be unavoidably generated during GTOA's solving process. At present, there are several common methods for handling infeasible solutions, such as the Penalty function approach, the Repair approach, the Purist approach, the Separation approach, and the Hybrid approach [44–49]. Existing studies show that the greedy strategy-based repair and optimization methods are the best fit for KPs [9,15,44,45]. According to the mathematical model of KPs and the greedy strategy-based repair and optimization method, we will present the efficient approaches to solving KPs by using GTOA.

3.1. Solving set union Knapsack problem by GTOA

The SUKP [7,50], belonging to the first category of KPs, has important application in investment decision-making, flexible manufacturing systems, the database system with primary, and secondary memories [3,7,51]. Goldschmidt et al. [7] first used

hyper graph to propose a deterministic algorithm for solving SUKP based on dynamic programming, but its time complexity is exponential, which leads to very poor practicability. By means of the d-regular graph theory, Ashwin Arulselvan [50] proposed an approximation algorithm

A-SUKP with the approximation ratio of $\overline{\left(1-e^{\frac{-1}{d}}\right)}$ based on the greedy

strategy, where $d(d \ge 2)$ is the supremum of the frequencies of all elements. A-SUKP greatly improved the speed for solving SUKP. Yichao He et al. [44] proposed a novel binary artificial bee colony (BABC) to solve SUKP and conducted a comparison with A-SUKP showing the BABC not only has a better performance but is also faster.

Definition of SUKP: Given a set of elements $\mathbf{U} = \{1,2,\dots,m\}$ and a set of items $\mathbf{S} = \{1,2,\dots,n\}$, where each item $i \in \mathbf{S}$ corresponds to a nonempty subset $\mathbf{U}_i \subseteq \mathbf{U}$ and has a profit $p_i > 0$; every element $j \in \mathbf{U}$ has a weight $w_j > 0$. For any nonempty subset $\mathbf{A} \subseteq \mathbf{S}$, the profit of \mathbf{A} is defined as $P(\mathbf{A}) = \sum_{i \in \mathbf{A}} p_i$, and the weight is $W(\mathbf{A}) = \sum_{j \in \bigcup_{i \in \mathbf{A}} \mathbf{U}_j} w_j \leq C$. For a given knapsack capacity C, The goal of SUKP is to find a subset of items $\mathbf{S}^* \subseteq \mathbf{S}$, such that $P(\mathbf{S}^*)$ is maximized and $W(\mathbf{S}^*) \leq C$.

Without loss of generality, we suppose that p_i (i = 1, 2, ..., n), w_j (j = 1, 2, ..., m) and *C* are positive integers, the set family $\{\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_n\}$ is a cover of **U**, where $\mathbf{U}_i \subseteq \mathbf{U}$ and $\mathbf{U}_i \neq \emptyset$; $W(\mathbf{S}) > C$, and for $i \in \mathbf{S}$, $\sum_{j \in U_i} w_j \leq C$. Let $X = (x_1, x_2, ..., x_n)$ be an *n*-dimensional 0–1 vector, and

Table 3

Comparison among GTOA, GA, BPSO and BABC for the 3rd category of SUKP.

Index	Instance	CBEST	Algorithm	Best	Mean	Worst	StD	Time
1	sukp85_100	12,045	GA	11454	11092.7	10749	171.22	0.113
	_0.10_0.75		BPSO	11710	11482.6	11174	189.17	0.190
			BABC	11664	11182.7	10897	183.57	0.188
			GTOA	12045	11388.3	11083	107.48	0.121
2	sukp85_100	12,369	GA	12124	11326.3	10369	417.00	0.131
	_0.15_0.85		BPSO	12369	11750.7	11374	424.40	0.211
			BABC	12369	12081.6	11710	193.79	0.217
			GTOA	12369	11945.7	11251	400.84	0.126
3	sukp185_200	13,696	GA	12841	12236.6	11843	198.18	1.231
	_0.10_0.75		BPSO	13497	12703.0	12247	382.50	1.572
			BABC	13047	12522.8	12011	201.35	1.502
			GTOA	13647	13143.5	12170	308.36	1.099
4	sukp185_200	11,298	GA	10920	10351.5	9832	208.08	1.204
	_0.15_0.85		BPSO	10920	10242.5	9783	373.53	1.732
			BABC	10602	10150.6	9900	152.91	1.948
			GTOA	10973	10566.1	9790	292.99	1.177
5	sukp285_300	11,568	GA	10994	10640.1	10304	126.84	3.827
	_0.10_0.75		BPSO	11538	11104.9	10419	190.23	5.612
			BABC	11158	10775.9	10584	116.80	5.450
			GTOA	11568	11202.4	10734	201.21	3.667
6	sukp285_300	11,763	GA	11093	10190.3	9737	249.76	3.990
	_0.15_0.85		BPSO	11377	10529.8	9767	320.93	6.844
			BABC	10528	9897.9	9622	186.53	5.571
			GTOA	11377	10821.9	10004	319.23	3.947
7	sukp385_400	10,326	GA	9799	9432.8	9137	163.84	9.325
	_0.10_0.75		BPSO	10252	9782.2	9089	222.64	12.985
			BABC	10085	9537.5	9202	184.62	13.012
			GTOA	10326	9949.3	9501	165.80	8.892
8	sukp385_400	10,302	GA	9173	8703.7	8342	154.15	9.911
	_0.15_0.85		BPSO	10302	9131.5	8198	271.05	14.953
			BABC	9456	9090.0	8694	156.69	13.724
			GTOA	10302	9381.3	8841	286.24	9.887
9	sukp485_500	11,037	GA	10311	9993.2	9799	117.73	18.708
	_0.10_0.75		BPSO	10923	10461.7	9929	232.03	26.590
			BABC	10823	10483.4	9965	228.34	27.227
			GTOA	11037	10658.0	10206	164.93	18.670
10	sukp485_500	9964	GA	9329	8849.5	8586	141.84	20.129
	_0.15_0.85		BPSO	9589	9180.1	8631	217.33	29.108
			BABC	9333	9085.6	8666	115.62	28.493
			GTOA	9964	9356.3	8785	205.87	18.896

 $A_X = \{i | x_i \in X \land x_i = 1, 1 \le i \le n\} \subseteq S$. For any i $(i = 1, 2, \dots, n)$, $x_i = 1$ if and only if $i \in A_X$. By the one-to-one relationship between X and A_X , the mathematical model of SUKP is represented as:

$$Max \ f(X) = \sum_{i=1}^{n} x_i p_i \tag{7}$$

s.t.
$$W(A_X) = \sum_{j \in \bigcup_{i \in A_X} U_i} w_j \le C$$
 (8)

When using GTOA to solve SUKP, every individual is an *n*-dimensional 0–1 vector on {0, 1}^{*n*}. It should be noted that any of the *n*-dimensional 0–1 vectors on {0, 1}^{*n*} is only a potential solution of SUKP, and it is a feasible solution only when satisfying the constraint (8), otherwise it is an infeasible solution. Yichao He et al. [44] proposed a repair and optimization algorithm S-GROA to eliminate the infeasible solutions when solving SUKP by EAs. In S-GROA, d_j ($j = 1, 2, \dots, m$) denotes the frequency of the element $j(j \in U)$ in subsets U_1, U_2, \dots, U_n , and $T_i = \sum_{j \in U_i} (w_j/d_j)$ ($i = 1, 2, \dots, n$). On the basis of using S-GROA to deal with infeasible solution, the algorithm for solving SUKP by GTOA can be described as follows:

Since the time complexity of the *SortAlgorithm* [12] and S-GROA [44] are O(nlogn) and O(nm), respectively, the time complexity of Algorithm 4 is $O(n^3m)$.

3.2. Solving discounted {0-1} Knapsack problem by GTOA

Guldan [5] first proposed D{0-1}KP and gave its heuristic and deterministic algorithms. Aiying Rong et al. [6] defined the kernel of D{0-1}KP and discussed the validity of solving D{0-1}KP by deterministic algorithm. Yichao He et al. [45] presented two evolutionary algorithms FirEGA and SecEGA for solving D{0-1}KP based on GA. Furthermore, they studied deterministic algorithms and approximation algorithms of D{0-1}KP in the literature [49], and note that PSO-GRDKP is a high performance evolutionary algorithm for solving D{0-1}KP.

Definition of D{0-1}KP [5,6]: Given a set of *n* item groups, suppose that each group $i(i = 0, 1, \dots, n - 1)$ consists of three items: 3i, 3i + 1 and 3i + 2. The item 3i has weight w_{3i} and profit p_{3i} , and the item 3i + 1 has weight $w_{3i + 1}$ and profit $p_{3i + 1}$. The first two items 3i and 3i + 1 are paired to derive the third item 3i + 2 with profit $p_{3i + 2} = p_{3i} + p_{3i + 1}$ and the discounted weight $w_{3i + 2}$, which satisfies $w_{3i + 2} < w_{3i} + w_{3i + 1}$, $w_{3i} < w_{3i + 2}$ and $w_{3i + 1} < w_{3i + 2}$. In each group, at most one of the three items can be selected to be placed in the knapsack with capacity *C*. The problem is how to select items loaded into the knapsack such that the total profit is maximized under the condition that the total weight of the selected items does not exceed *C*.

Table 4

Comparison among FirEGA, SecEGA, GPSO and GTOA for UDKP1-UDKP10.

Index	Instance	OPT	Algorithm	Best	Mean	Worst	StD	Gap
1	UDKP1	85,740	FirEGA	80593	79103.2	77935	690.01	7.740
			SecEGA	78,287	76807.2	75156	798.95	10.420
			GPSO	85740	85669.3	85459	67.511	0.080
			GOTA	85740	85684.8	85489	53.8687	0.064
2	UDKP2	163,744	FirEGA	155039	151662	149875	1044.95	7.380
			SecEGA	148043	145548	143833	883.434	11.110
			GPSO	163744	163710	163566	33.076	0.020
			GOTA	163744	163727	163640	23.1604	0.010
3	UDKP3	269,393	FirEGA	246698	240886	237980	1491.97	10.580
			SecEGA	228823	225492	222486	1353.58	16.300
			GPSO	269340	269124	268504	129.523	0.100
			GOTA	269393	269305	269148	57.4767	0.033
4	UDKP4	347,599	FirEGA	321605	317319	314486	1426.85	8.710
			SecEGA	305796	299978	297606	1435.46	13.700
			GPSO	347541	347267	346786	147.994	0.100
			GOTA	347582	347535	347462	27.6553	0.018
5	UDKP5	442,644	FirEGA	405409	399620	395367	1692.23	9.720
			SecEGA	376147	370808	367574	1611.71	16.230
			GPSO	441693	440555	439151	464.361	0.479
			GOTA	442605	442514	442332	51.4972	0.029
6	UDKP6	536,578	FirEGA	486556	478726	474015	2233.61	10.780
			SecEGA	447438	442499	438809	1765.28	17.530
			GPSO	534571	532997	531429	707.305	0.670
			GOTA	536563	536177	535573	215.415	0.075
7	UDKP7	635,860	FirEGA	568119	560948	556938	2441.8	11.780
			SecEGA	529753	521401	518407	1813.04	18.000
			GPSO	632919	631497	629352	746.345	0.690
			GOTA	635311	634648	633941	323.171	0.191
8	UDKP8	650,206	FirEGA	590137	585286	580684	2078.87	9.980
			SecEGA	550645	546678	543836	1449.36	15.920
			GPSO	646602	644282	641659	877.06	0.910
			GOTA	649514	648719	647259	416.966	0.229
9	UDKP9	718,532	FirEGA	655172	649636	645012	2023.64	9.590
			SecEGA	613581	602215	605835	2003.75	16.190
			GPSO	712591	710039	707289	967.617	1.180
			GOTA	717243	715941	714110	533.289	0.361
10	UDKP10	779,460	FirEGA	712270	706575	701545	2013.43	9.350
			SecEGA	665459	658908	655645	1723.8	15.470
			GPSO	773678	771246	768946	1027.4	1.050
			GOTA	778576	777781	776820	370.856	0.215

Without losing generality, we suppose that p_j , w_j ($0 \le j \le 3n - 1$), and *C* are positive integers, $w_{3i+2} \le C$ ($0 \le i \le n - 1$), and $\sum_{i=0}^{n-1} w_{3i+2} > C$. Two mathematical models of D{0-1}KP are represented as follows.

The first mathematical model [5,6]:

$$Max \ f(X) = \sum_{i=0}^{n-1} \left(x_{3i}p_{3i} + x_{3i+1}p_{3i+1} + x_{3i+2}p_{3i+2} \right)$$
(9)

s.t.
$$x_{3i} + x_{3i+1} + x_{3i+2} \le 1, i = 0, 1, \dots, n-1,$$
 (10)

$$\sum_{i=0}^{n-1} \left(x_{3i} w_{3i} + x_{3i+1} w_{3i+1} + x_{3i+2} w_{3i+2} \right) \le C, \tag{11}$$

$$x_{3i}, x_{3i+1}, x_{3i+2} \in \{0, 1\}, i = 0, 1, \cdots, n-1.$$
(12)

where $X = (x_0, x_1, \dots, x_{3n-1}) \in \{0, 1\}^{3n}$, $x_j (0 \le j \le 3n - 1)$ is used to indicate whether the item *j* is loaded into knapsack. The item *j* is loaded into knapsack if and only if $x_j = 1$.

The second mathematical model [45]:

$$Max \ f(X) = \sum_{i=0}^{n-1} \left\lceil \frac{x_i}{3} \right\rceil p_{3i+|x_i-1|}$$
(13)

s.t.
$$\sum_{i=0}^{n-1} \left\lceil \frac{x_i}{3} \right\rceil w_{3i+|x_i-1|} \le C$$
 (14)

$$x_i \in \{0, 1, 2, 3\}, i = 0, 1, \cdots, n-1.$$
 (15)

where $X = (x_0, x_1, \dots, x_{n-1}) \in \{0, 1, 2, 3\}^n.$ The integer variable $x_i (0 \le i \le n - 1)$ indicates whether there is an item of the item group *i* to be loaded into the knapsack. $x_i = 0$ indicates that no items of item group *i* is loaded into the knapsack; $x_i = 1$ expresses that the item 3*i* is loaded into the knapsack; $x_i = 2$ indicates that the item 3i + 1 is loaded into the knapsack; $x_i = 3$ expresses that the item 3i + 2 is loaded into the knapsack. [x] is top function of x representing the smallest integer not less than x. Obviously, the length of the feasible solution in the second mathematical model of D{0-1}KP is 1/3 that of the first one. Therefore, when use GTOA to solve D{0-1}KP under the second mathematical model, it can not only save storage space but also improve the algorithm's running speed.

To eliminate the infeasible solutions of D{0-1}KP, we refer to the skill in [49] to propose a repair and optimization algorithm D-GROA. Similar to SUKP, sort all items of the D{0-1}KP instance according to the order of p_j/w_j ($0 \le j \le 3n - 1$) from large to small, and store the subscripts of the items into the array $H[0\cdots 3n - 1]$ according to the order. Let $X = (x_0, x_1, \dots, x_{n-1}) \in \{0, 1, 2, 3\}^n$ be a potential solution of the

Table 5

Comparison among FirEGA, SecEGA, GPSO and GTOA for WDKP1-WDKP10.

Index	Instance	OPT	Algorithm	Best	Mean	Worst	StD	Gap
1	WDKP1	83,098	FirEGA	82803	82693.2	82592	52.0424	0.487
			SecEGA	80014	79021.8	78096	473.674	4.905
			GPSO	83098	83086.5	83058	6.85054	0.014
			GOTA	83098	83083.9	83058	7.0420	0.017
2	WDKP2	138,215	FirEGA	137704	137584	137356	63.228	0.457
			SecEGA	133315	132276	131337	415.619	4.297
			GPSO	138215	138202	138133	18.2569	0.009
			GOTA	138215	138202	138157	11.3193	0.009
3	WDKP3	256,616	FirEGA	254120	253657	253307	173.011	1.153
			SecEGA	238331	235721	234025	873.581	8.143
			GPSO	256616	256573	256493	24.6037	0.017
			GOTA	256616	256605	256575	9.3001	0.004
4	WDKP4	315,657	FirEGA	313966	312849	311998	484.755	0.890
			SecEGA	293640	290851	288764	950.06	7.859
			GPSO	315653	315605	315493	32.0495	0.017
			GOTA	315657	315635	315596	12.9266	0.007
5	WDKP5	428,490	FirEGA	426311	424548	423058	798.53	0.920
			SecEGA	393617	390014	387992	1059.83	8.979
			GPSO	428484	428419	428303	34.5386	0.017
			GOTA	428487	428469	428431	12.6941	0.005
6	WDKP6	466,050	FirEGA	463185	461672	457718	1107.57	0.939
			SecEGA	429208	425112	423269	1058.37	8.784
			GPSO	466019	465947	465828	45.2202	0.022
			GOTA	466050	466036	466008	8.3908	0.003
7	WDKP7	547,683	FirEGA	544019	541949	538126	1224.68	1.047
			SecEGA	501557	496134	493845	1230.94	9.412
			GPSO	547565	547355	547138	87.7251	0.060
			GOTA	547675	547647	547572	17.000	0.007
8	WDKP8	576,959	FirEGA	573427	571559	563253	1495.36	0.936
			SecEGA	530971	523203	520350	2157.09	9.317
			GPSO	576800	576597	576339	87.4878	0.063
			GOTA	576954	576904	576820	28.4346	0.010
9	WDKP9	650,660	FirEGA	647477	644820	630086	2056.06	0.898
			SecEGA	598343	586770	583854	2315.5	9.819
			GPSO	650502	650259	649938	107.902	0.062
			GOTA	650648	650596	650450	33.5694	0.010
10	WDKP10	678,967	FirEGA	675452	673008	668239	1441.96	0.878
			SecEGA	620230	606215	609964	3090.86	10.715
			GPSO	678862	678662	678401	91.1195	0.045
			GOTA	678945	678857	678551	63.5837	0.016

j

D{0-1}KP instance; $\lfloor x \rfloor$ is the bottom function representing the maximum integer not greater than *x*; *H*[*j*](*mod*3) denotes the remainder when *H*[*j*] is divided by 3. Then, the pseudo-code of D-GROA is described as follows:

In D-GROA, $f(X) = \sum_{i=0}^{n-1} \left[\frac{x_i}{3} \right] p_{3i+|x_i-1|}$ is used as the fitness of X,

which is obtained in Step 13. The time complexity of D-GROA is O(n). On the basis of using D-GROA to eliminate infeasible solutions, the

algorithm for solving D $\{0-1\}$ KP by GTOA is given as follows.

Obviously, the time complexity of Algorithm 6 is $O(n^3)$.

3.3. Solving the bounded Knapsack problem by GTOA

The BKP [3] belongs to the third category of KPs. It is defined as follows. Let $N = \{1, 2, \dots, n\}$ be a set of n items to be packed into a knapsack of size C. For $j = 1, 2, \dots, n$, let p_j w_j and b_j be the profit, weight and the number of identical copies available for the *j*th item, respectively, where C and p_j , w_j , b_j are all positive integers. The objective is to fill the knapsack with some items from N whose total weight is at most C and such that their total profit is maximized. The mathematical model of BKP is shown as follows.

$$Max \ f(X) = \sum_{j=1}^{n} p_j x_j \tag{16}$$

$$s.t \quad \sum_{j=1}^{n} w_j x_j \le C \tag{17}$$

$$0 \le x_j \le b_j, x_j \text{ is integer}, j = 1, 2, \cdots, n.$$
(18)

where $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}[b_1 + 1, b_2 + 1, \dots, b_n + 1]$. None of the copies of the *j*th item is loaded into the knapsack when $x_j = 0$, otherwise x_j copies of the *j*th item are loaded. Obviously, 0-1KP is a special case of BKP when $b_j = 1(j = 1, 2, \dots, n)$. Without loss of generality, suppose there is at least one *j* which satisfies $b_j \ge 2$, $Max\{b_jw_j|1 \le j \le n\} \le C$ and $\sum_{i=1}^n b_j w_j > C$.

Similar to SUKP and D{0-1}KP, we give a repair and optimization algorithm B-GROA to handle infeasible solution: Sort all the items in the BKP instance according to the order of p_j/w_j ($1 \le j \le n$) from large to small, and store the subscripts sequentially into the array $H[1\cdots n]$ based on the order. Let $X = (x_1, x_2, \cdots, x_n) \in \mathbb{Z}[b_1 + 1, b_2 + 1, \cdots, b_n + 1]$ denote a potential solution of BKP. The pseudo-code of B-GROA is described as below.

Obviously, the time complexity of B-GROA is O(n).

Comparison among FirEGA, SecEGA, GPSO and GTOA for SDKP1-SDKP10.

Index	Instance	OPT	Algorithm	Best	Mean	Worst	StD	Gap
1	SDKP1	94,459	FirEGA	93235	93170.8	93070	42.147	1.365
			SecEGA	89769	88831.5	87463	594.911	5.958
			GPSO	94449	94260.7	93818	110.235	0.210
			GOTA	94452	94425.5	94329	22.8099	0.035
2	SDKP2	160,805	FirEGA	159159	159004	158859	61.538	1.120
			SecEGA	153821	152059	150753	489.391	5.439
			GPSO	160777	160607	160253	95.830	0.123
			GOTA	160805	160769	160701	19.9885	0.022
3	SDKP3	238,248	FirEGA	235454	235241	235043	79.858	1.262
			SecEGA	224997	223580	221918	543.378	6.157
			GPSO	238158	237900	237606	99.133	0.146
			GOTA	238243	238204	238144	21.1418	0.018
4	SDKP4	340,027	FirEGA	336353	335963	335709	122.410	1.1952
			SecEGA	318510	315513	313747	851.135	7.2094
			GPSO	339830	339526	339156	131.691	0.1473
			GOTA	340025	339995	339933	17.6036	0.009
5	SDKP5	463,033	FirEGA	452900	447587	444255	1974.990	3.3368
			SecEGA	420238	416964	413933	1291.650	9.949
			GPSO	462107	461566	460906	260.874	0.317
			GOTA	462980	462868	462717	48.4461	0.036
6	SDKP6	466,097	FirEGA	459254	458893	458584	162.938	1.546
			SecEGA	430738	427304	425504	1031.120	8.323
			GPSO	465378	464856	464171	222.960	0.266
			GOTA	466074	466010	465881	33.8705	0.019
7	SDKP7	620,446	FirEGA	599361	592279	579673	3949.030	4.540
			SecEGA	561224	556083	552007	1926.260	10.374
			GPSO	618753	617827	616602	342.714	0.422
			GOTA	620219	619978	619623	109.375	0.075
8	SDKP8	670,697	FirEGA	661276	660104	659367	426.056	1.579
			SecEGA	611644	606263	603774	1446.940	9.607
			GPSO	668821	668107	667341	322.923	0.386
			GOTA	670304	670071	669791	124.218	0.093
9	SDKP9	739,121	FirEGA	729135	727544	727064	343.670	1.566
			SecEGA	674885	667900	664580	1614.040	9.636
			GPSO	736589	735805	734871	349.522	0.449
			GOTA	738458	738107	737667	179.945	0.137
10	SDKP10	765,317	FirEGA	756205	753394	750757	985.464	1.558
			SecEGA	708935	695557	691994	2956.08	9.115
			GPSO	762603	761980	761258	288.412	0.436
			GOTA	764586	764159	763609	197.927	0.151

When solving BKP by GTOA, we apply B-GROA to eliminate infeasible solution, which is similar to the procedure in SUKP and D $\{0-1\}$ KP. To avoid duplication, we omit it.

4. Experimental results and comparative studies

As we all know, GA [17,46] is one of the most commonly used classic EAs, suitable for solving combinatorial optimization problems. BPSO [36] is a simple and effective EAs, and performs excellently when solving D{0-1}KP [49] and RTVKP [52]. Literature [44] is the only study that uses EAs to solve SUKP, in which the proposed BABC performs better than GA [17,46], binDE [35], ABCbin [38], and A-SUKP [50]. To verify the performance of GTOA for solving KPs, we experimentally conduct a comparison between GTOA and GA, BPSO, BABC, FirEGA [45], and SecEGA [45] for their computational performance.

All experiments are were carried out on Acer Aspire E1-570G with Intel(R) Core(TM)i5-3337u CPU-1.8GHz, 4GB DDR3 (3.82GB available); the operating system is Microsoft Windows 8. All algorithms are implemented using C + +, and the compilation environment is Visual C + + 6.0. The figures are drawn by the MATLAB7.10.0.499 (R2010a).

4.1. Simulation results for the SUKP instances

4.1.1. Parameters setting of the algorithms

To determine the optimal value of P_m in GTOA, we solve three instances *sukp*300_285_0.10_0.75, *sukp*300_300_0.10_0.75, and *sukp*285_300_0.10_0.75 [44] by GTOA when P_m =0.005,0.006,0.007,0.008,0.009,0.010, and 0.011. Each instance is independently calculated 100 times. According to the box-plots in Figs. 1–3, it is easy to see that $0.008 \le P_m \le 0.009$ is the most reasonable choice.

When using GTOA,GA,BPSO,and BABC to solve three kinds of large-scale SUKP instances [44], the population size of all algorithms is set as NP = 20, individual encoding is an *n*-dimensional 0–1 vector, the iteration number is set as $MIT = Max\{m,n\}$, *n* is the number of items and *m* is the number of elements, S-GROA [44] is used to deal with infeasible solutions. In GTOA, *F* is a random 0–1 vector, $P_m = 0.008$. In GA [46], the single point crossover operator, the uniform mutation operator, and the fitness proportional model are used; the crossover probability is $P_c = 0.8$ and the mutation probability is $P_m = 0.01$. In BPSO [36], set W = 1.0, $C_1 = C_2 = 2.0$, and each dimensional component in the *n*-dimensional real vector is in [-5.0,5.0]. In BABC [44], a = 5.0 and the threshold value is *limit* = $Max\{m,n\}/5$.

Comparison among FirEGA ,SecEGA, GPSO and GTOA for IDKP1-IDKP10.

Index	Instance	OPT	Algorithm	Best	Mean	Worst	StD	Gap
1	IDKP1	70,106	FirEGA	70106	70099.4	70090	7.234	0.009
			SecEGA	68663	67999.8	67369	328.441	3.004
			GPSO	70106	70098.1	70077	6.685	0.011
			GOTA	70106	70089.3	70048	15.1772	0.024
2	IDKP2	118,268	FirEGA	118169	117869	117625	102.598	0.337
			SecEGA	114434	113385	112307	446.672	4.129
			GPSO	118268	118253	118202	15.687	0.013
			GOTA	118268	118239	118168	23.6627	0.025
3	IDKP3	234,804	FirEGA	234497	233997	233666	175.423	0.344
			SecEGA	220096	217982	216313	835.828	7.164
			GPSO	234804	234796	234759	9.202	0.003
			GOTA	234804	234792	234731	12.5576	0.005
4	IDKP4	282,591	FirEGA	282148	280695	278881	827.625	0.671
			SecEGA	263238	260425	258922	933.402	7.844
			GPSO	282591	282578	282554	9.553	0.005
			GOTA	282,583	282560	282497	16.2573	0.011
5	IDKP5	335,584	FirEGA	335004	333484	329621	1173.900	0.626
			SecEGA	309573	306878	304881	907.187	8.554
			GPSO	335584	335580	335546	7.344	0.001
			GOTA	335,580	335549	335449	27.8969	0.010
6	IDKP6	452,463	FirEGA	451680	449863	446704	1161.520	0.575
			SecEGA	414090	411367	408788	1099.310	9.083
			GPSO	452463	452452	452425	9.168	0.002
			GOTA	452463	452438	452372	18.5076	0.006
7	IDKP7	489,149	FirEGA	488009	485592	476385	2294.280	0.727
			SecEGA	451528	444316	442133	1280.310	9.166
			GPSO	489149	489133	489105	8.268	0.003
			GOTA	489,144	489109	489019	23.9663	0.008
8	IDKP8	533,841	FirEGA	533035	529984	514196	2308.110	0.722
			SecEGA	490494	481831	478035	2215.660	9.743
			GPSO	533839	533827	533808	6.246	0.003
			GOTA	533,834	533807	533762	14.4209	0.006
9	IDKP9	528,144	FirEGA	526410	523982	511651	2216.130	0.788
			SecEGA	489661	477001	471848	3656.220	9.684
			GPSO	528140	528131	528094	9.466	0.002
			GOTA	528144	528103	527998	28.6709	0.008
10	IDKP10	581,244	FirEGA	578903	576772	568903	1905.180	0.769
			SecEGA	535541	521604	516445	4265.070	10.261
			GPSO	581244	581230	581194	10.610	0.002
			GOTA	581,234	581200	581086	24.9040	0.008

4.1.2. Calculation and comparison

In Tables 1–3 (in Appendix), *CBEST* is the best result currently known for the instance. *Best* and *Worst* are the best value and the worst value, respectively, obtained by GTOA, GA, BPSO, and BABC when solving every instance 100 times independently. *Mean* and *StD* denote the mean value and the standard deviation, respectively, and *Time* is the average running time that each algorithm takes for individually solving every instance.

Tables 1–3 illustrate that, among the 30 instances, GTOA can achieve the best result currently known of 22 instances, BPSO does that on 5 instances, BABC does that on 2 instances, while GA gets the best result currently known on only one instance. Regarding the average running time, the solving speeds of GA and GTOA are almost equal and obviously faster than those of BPSO and BABC. The difference between the speeds of BABC and BPSO is small.

Since EAs are a type of stochastic approximation algorithms, in order to evaluate its performance, we also need to consider the statistical features of the average performance and stability of all algorithms. Comparisons on the average performance of all algorithms can be conducted by using the *Gap* fitting curves, where the *Gap* metric is the relative difference between the optimal values *OPT* and the mean value *Mean*, whose formulation is given in (19). The closer the *Gap* curve is to the abscissa axis, the better the mean performance of the algorithm is. In addition, we can draw a histogram according to the value of *StD* and evaluate the stability of all algorithms by the distribution of columns.

$$Gap = \frac{|OPT - Mean|}{OPT} \times 100\,(\%) \tag{19}$$

Since *OPT* of SUKP instances is unknown, *OPT* is replaced by *CBEST* when calculating the *Gap* of GTOA, GA,BPSO,and BABC. The *Gap* curves for each algorithm are given in Figs. 4–6, and the histograms of *StD* are drawn in Figs. 7–9.

From Figs. 4–6, we can see that the performance of GTOA is best in the four algorithms since its average performance is much better than those of GA, BPSO, and BABC; BPSO and BABC rank second; and GA is the worst. Figs. 7–9 show that stability of all algorithms are basically equal.

4.2. Simulation results for the D{0-1}KP instances

4.2.1. Parameters setting of the algorithms

Using the same method as Section 4.1.1, we find that $P_m = 0.008$ is a suitable value when solving D{0-1}KP by GTOA.

FirEGA and SecEGA [45] are the most classic two EAs for solvingD{0-1}KP based on GA and different mathematical models. PSO-GRDKP[49] is the most efficient evolutionary al-

Table 8 Comparisons between GA and GTOA for solving instances UBKP1-UBKP10.

		GA				GTOA			
Instance	OPT	Best	Mean	Worst	StD	Best	Mean	Worst	StD
UBKP1	201616	201616	201582.0	201499	26.904	201616	201614.0	201590	6.361
UBKP2	414114	414114	414107.0	414028	22.857	414114	414114	414114	0.0
UBKP3	594613	594610	594597.0	594574	9.181	594613	594608.0	594587	3.892
UBKP4	831629	831629	831612.0	831599	6.890	831629	831628.0	831611	4.107
UBKP5	1003643	1003642	1003632.7	1003628	4.249	1003643	1003640.2	1003626	3.820
UBKP6	1228085	1228085	1228080.5	1228073	3.220	1228085	1228079.7	1228073	2.965
UBKP7	1524770	1524770	1524764.8	1524755	5.717	1524770	1524769.0	1524759	3.056
UBKP8	1692853	1692846	1692838.5	1692829	4.493	1692853	1692847.3	1692835	4.917
UBKP9	1869142	1869138	1869133.5	1869126	3.025	1869142	1869134.8	1869127	3.385
UBKP10	2,066,060	2066060	2066057.2	2066055	1.817	2066060	2066058.2	2066052	2.559

Table 9

_,,				11017	2000000		2000002	21009
een GA and GTOA for solvi	ng instances WBKP1–WBKI	P10.		-5				
	GA				GTOA			
OPT	Best	Mean	Worst	StD	Best	Mean	Worst	StD
119312	119312	119305.0	119288	5.293	119312	119310.0	119308	1.521
297700	297700	297700	297700	0.0	297700	297699.8	297687	1.566
444156	444156	444150.0	444142	4.328	444156	444151.0	444142	3.593
605678	605678	605671.0	605664	2.917	605678	605676.0	605671	1.802
772191	772191	772188.0	772183	0.980	772191	772186.0	72183	1.708
890314	890313	890311.0	890309	0.774	890313	890312.0	890310	0.989
1045302	1045302	1045299.0	1045298	0.942	1045300	1045298.5	1045296	1.284
1210947	1210946	1210944.1	1210944	0.276	1210945	1210944.6	1210940	0.984
1407365	1407365	1407364.0	1407361	0.374	1407365	1407364.1	1407364	0.099
1,574,079	1574079	1574076.4	1574073	1.278	1574079	1574077.6	1574073	1.538
	OPT 119312 297700 444156 605678 772191 890314 1045302 1210947 1407365 1,574,079	GA GA OPT Best 119312 119312 297700 297700 444156 444156 605678 605678 772191 772191 890314 890313 1045302 1045302 1210947 1210946 1407365 1407365 1,574,079 1574079	GA GA OPT Best Mean 119312 119312 119305.0 297700 297700 297700 444156 444156 444150.0 605678 605678 605671.0 772191 772188.0 890313 890311.0 1045302 1045302 1045299.0 1210947 1210944.1 1407365 1407365 1407364.0 1,574,079 1574076.4	GA Mean Worst 119312 119312 119305.0 119288 297700 297700 297700 297700 444156 444156 444150.0 444142 605678 605678 6056674 605664 772191 772191 772188.0 772183 890314 890313 890311.0 890309 1045302 1045299.0 1045298 1210944 1407365 1407365 1407364.0 1407361 1,574,079 1574079 1574076.4 1574073	Arrow Mathematical Arrow Mathmatecal Arrow Mathematical Arrow Mathematical Arrow Mat	Annu Kanada GTOA for solving instances WBKP1-WBKP10. GA GTOA OPT Best Mean Worst StD Best 119312 119312 119305.0 119288 5.293 119312 297700 297700 297700 297700 297700 444156 444156 444156 444156 605678 605678 605671.0 605664 2.917 605678 772191 772191 772188.0 772183 0.980 772191 890314 890313 89031.0 890309 0.774 890313 1045302 1045302 1045299.0 1045298 0.942 1045300 1210947 1210946 1210944.1 1210944 0.276 1210945 1407365 1407365 1407364.0 1407361 0.374 1407365 1,574,079 1574076.4 1574073 1.278 1574079	GA GA GTOA GA GTOA GPT Best Mean Worst StD Best Mean 119312 119312 119305.0 119288 5.293 119312 119310.0 297700 297700 297700 297700 297700 297699.8 444156 444156 444156.0 444142 4.328 444156 444151.0 605678 605678 605671.0 605664 2.917 605678 605676.0 772191 772188.0 772183 0.980 772191 772186.0 890314 890313 890311.0 890309 0.774 890313 890312.0 1045302 1045299.0 1045298 0.942 1045300 1045298.5 1210947 1210946 1210944.0 0.276 1210945 1210944.6 1407365 1407365 1407365 1407365 1407365 1407365 1407365 1407365 1407365 1407365 1407365 1	GA and GTOA for solving instances WBKP1-WBKP10. GTOA GTOA OPT Best Mean Worst StD Best Mean Worst 119312 119312 119305.0 119288 5.293 119312 119310.0 119308 297700 297700 297700 0.0 297700 297687 444156 444156 444150.0 444142 4.328 444156 444151.0 444142 605678 605678 605677.0 605664 2.917 605678 605676.0 605671 772191 772191 772188.0 772183 0.980 772191 772186.0 72183 890314 890313 890311.0 890399 0.774 890313 890312.0 890310 1045302 1045292.0 1045298 0.942 1045300 1045298.5 1045296 1210947 1210946 1210944.1 1210944 0.276 1210945 1210944.6 1210940

Table 10

Comparison between GA and GTOA for solving instances SBKP1-SBKP10.

		GA				GTOA			
Instance	OPT	Best	Mean	Worst	StD	Best	Mean	Worst	StD
SBKP1	144822	144822	144818.0	144807	3.752	144822	144813.0	144799	6.135
SBKP2	259853	259853	259852.0	259848	1.050	259853	259852.0	259832	2.974
SBKP3	433414	433414	433414.0	433411	0.490	433414	433414.0	433413	0.099
SBKP4	493847	493847	493847.0	493846	0.218	493847	493847.0	493842	0.497
SBKP5	688246	688246	688246	688246	0.0	688246	688246.0	688239	0.829
SBKP6	849526	849526	849526	849526	0.0	849526	849518.0	849512	3.457
SBKP7	1060106	1060106	1060106	1060106	0.0	1060106	1060097.0	1060094	2.786
SBKP8	1171576	1171576	1171576	1171576	0.0	1171576	1171570.2	1171566	4.430
SBKP9	1263609	1263609	1263609	1263609	0.0	1263609	1263605.2	1263599	4.434
SBKP10	1412095	1412095	1412095	1412095	0.0	1412095	1412094.9	1412089	0.611

gorithm for solving D{0-1}KP currently, which is proposed by using BPSO (we denote PSO-GRDKP as GPSO for convenience). In the following, we will use GTOA, FirEGA, SecEGA, and GPSO to solve four kinds of large-scale D{0-1}KP instances [45]: UDKP1–UDKP10, WDKP1–UD-KP10, SDKP1–UDKP10, IDKP1–UDKP10, and evaluate the performance of GTOA by the comparative results.

FirEGA and GPSO use GR-DKP [49] to deal with infeasible solutions. GTOA and SecEGA apply D-GROA to handle infeasible solutions. The iteration number of all algorithms is set as $MIT = 10^*n$, where *n* is the number of item groups. In GTOA, the population size is set as NP = 20; $P_m = 0.008$. The population size of FirEGA and SecEGA are set as NP = 50; the others parameters are set to be equal to those in the literature [45]. In GPSO [49], the population size is set as NP = 20; each dimensional component in the 3*n*-dimensional real vector takes value in [-5.0, 5.0]; W = 1.0 and $C_1 = C_2 = 2.0$.

4.2.2. Calculation and comparison

All experimental results are based on 100 independent runs. In Tables 4–7 (in Appendix), *OPT* is the optimal value; *Best* and *Worst* represent the best and worst values among 100 calculated results respectively, and *Mean* and *StD* are the expectation and the standard deviation, respectively; *Gap* is the relative differences between *Mean* and *OPT* for each instance, which can be calculated by (19).

From the *Gap* in Tables 4–7 we can know that the *Gap* metrics for all algorithms are no more than 18.00, which illustrates that it is not only feasible but also highly efficient to solve D{0-1}KP by using GTOA, FirEGA, SecEGA, and GPSO. Furthermore, the comparison between the *Best* and *Worst* of all algorithms shows that, except the instance IDKP1, the *Worst* obtained by GTOA and GPSO compared to the other 39 instances are better than the *Best* gained by FirEGA and SecEGA, which indicates that the performances of GTOA and GPSO for solving D{0-1}KP are much better than those of FirEGA and SecEGA. Therefore, we only need to compare GTOA with GPSO in terms of the average performance and the stability to verify the efficient performance of GTOA.

The *Gap* fitting curves of GTOA and GPSO in Figs. 10–13 show that the maximum value of *Gap* for GTOA is no more than 0.4 and that for GPSO it is no more than 1.2, which illustrates that the average performances of both algorithms for solving D{0-1}KP are excellent. Moreover, for UDKP, WDKP and SDKP, it is obvious that the average performance of GTOA is much better than that of GPSO, but for IDKP, GPSO is slightly better than GTOA.

It can be known from the *StD* histograms of GTOA and GPSO in Figs. 14–17 that for UDKP, WDKP and SDKP, the algorithm stability of GTOA is better than that of GPSO, while for IDKP, GPSO is better.

The above comparative results indicate that the performances of both GTOA and GPSO are much better than those of FirEGA and SecEGA. Moreover, for UDKP, WDKP, and SDKP, GTOA has the best averaged performance and stability. It confirms that solving D{0-1}KP by GTOA is not only feasible but also highly efficient.

4.3. Simulation results for BKP instances

Among the current EAs, GA [17,46] is the only algorithm suitable for solving BKP since the feasible solutions of BKP are denoted as integer vectors in $Z[b_1 + 1, b_2 + 1, \dots, b_n + 1]$, where $b_j(1 \le j \le n)$ is a positive integer. Therefore, we verify

the performance of GTOA by experimentally comparing GTOA with GA for solving large-scale BKP instances.

So far there has not yet been a report of benchmark for BKP. We generate three kinds of large-scale BKP instances according to the method in [3].

- Uncorrelated instances of BKP(UBKP): *p_j* and *w_j* are randomly distributed in [1, *R*], 1≤*j*≤*n*.
- (2) Weakly correlated instances of BKP (WBKP): w_j randomly distributed in [1, *R*], and p_j randomly distributed in $\left[w_j \frac{R}{10}, w_j + \frac{R}{10}\right]$ such that $p_i \ge 1$.
- (3) Strongly correlated instances of BKP (SBKP): w_j randomly distributed in [1, *R*] and $p_j = w_j + 10$.

In all instances, R = 1000 and $C = 0.55 * \sum_{j=1}^{n} w_j b_j$, where $b_j(1 \le j \le n)$ is randomly distributed in interval [1,9]. For more details of the BKP instances, please refer to http://xxgc.hgu.edu.cn/uploads/heyichao/ThreekindsofBKPInstances.rar.

When using GA and GTOA to solve BKP, the iteration number is set as MIT = 2n, *n* is the number of items, and we use B-GROA to eliminate infeasible solutions. In GTOA, the population size is NP = 20, and we use RLCO and IRMO to generate new individuals; $P_m = 0.001$. In GA, the population size is NP = 50; we use single-point crossover, uniform mutation operators, and fitness proportional model to generate new individuals; $P_c = 0.8$ and $P_m = 0.001$.

In Tables 8–10 (in Appendix), *OPT* is the optimal value of instance; *Best* and *Worst* represent the best and worst values, respectively, obtained by using GTOA and GA in 100 independent runs, and *Mean* and *StD* are the mean value and standard deviation, respectively.

Tables 8–10 illustrate that in terms of *Best* and *Worst*, the absolute error |*OPT* – *Worst*| between *Worst* and *OPT* is no more than 117. In addition, for 4/5 of all BKP instances, the *OPT* can be attained by both algorithms, which indicates that the average performances of both GA and GTOA are excellent. Noting that the *StD* values of both GA and GTOA are no more than 27, we believe that both of GA and GTOA have very good stability.

Although both GA and GTOA perform well for solving BKP, the indexes in Tables 8–10 show that GTOA performs better when GTOA is used to solve the UBKP instances, and GA is better than GTOA for the SBKP instances. For the WBKP instances, both algorithms perform similarly. In summary, GTOA and GA have the same performance for solving BKP. They can efficiently solve BKP.

5. Conclusion

This paper presents an algebraic approach for designing evolutionary algorithms and proposes a new evolutionary algorithm GTOA by using the direct product of groups. To verify GTOA's effectiveness and wide range of applications, we apply it to solve different KPs, i.e., SUKP, D{0-1}KP, and BKP. In comparison with GA,BPSO,GPSO,BABC,FirEGA,and SecEGA for large-scale instances of KPs, we conclude that GTOA is not only easy to implement but also performs much better.

Some remarks and further investigations in this topic are listed: First, using the operations of a group to implement evolution search is a new and feasible evolution search mechanism. It is not only the innovation of algorithm design method but is also a successful paradigm for designing evolutionary algorithms by using the strict algebra theory. Second, the proposed GTOA is suitable not only for KPs but also for the others combinatorial optimization problems, for example, the satisfiability problem [27], the set cover problem [28,53], the decision making problem [54,55], and the machine scheduling problem [56]. Therefore, using GTOA to solve others combinatorial optimization problem is an important research topic. Finally, how to use GTOA to solve numerical optimization problems [57], topology optimization problem [58–60], feature selection problem [61], and semi-supervised clustering [62] is another topic which is worth researching in the future.

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Appendix

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